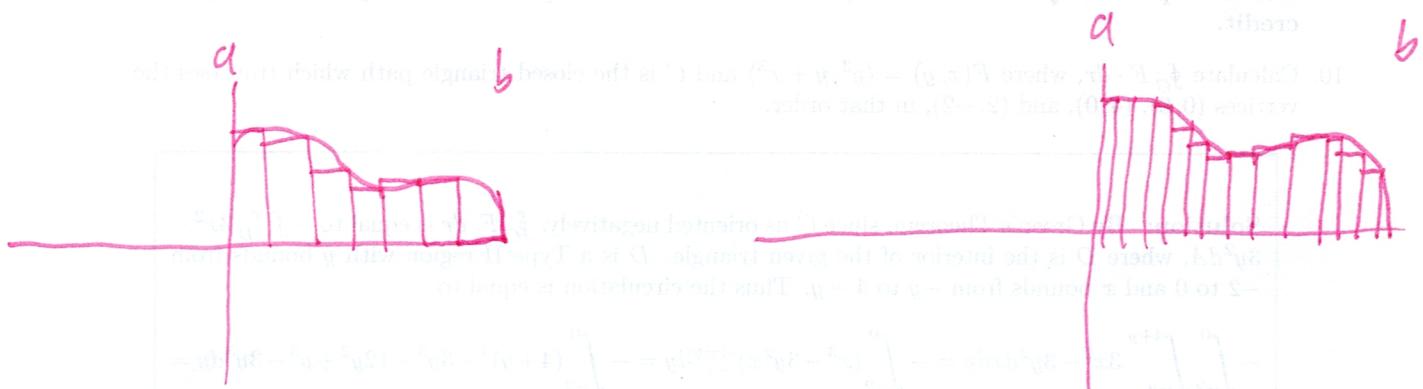


Iterated Integrals

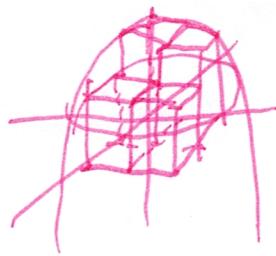
Recall from Calc 1 that we defined integrals using Riemann Sums;



$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{b-a}{n} \right) f\left(a + \left(\frac{b-a}{n}\right)i\right) = \lim_{\Delta x \rightarrow 0} \sum_{j=1}^n \Delta x f(x_j), \quad x_j = a + j \cdot \Delta x, \quad \Delta x = \frac{b-a}{n}$$

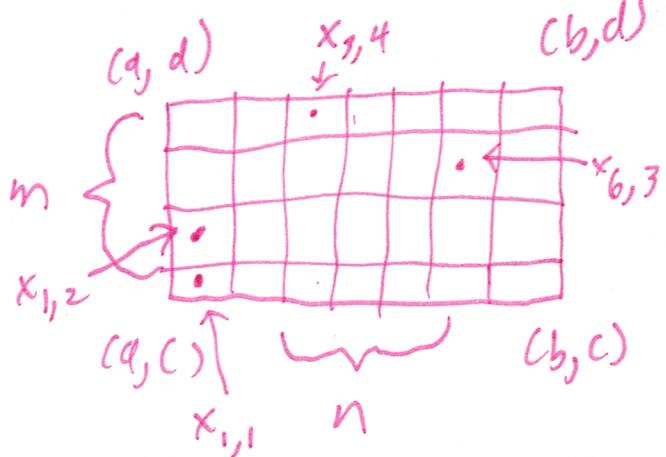
We do the same for two-variable functions:

$$R = [a, b] \times [c, d]$$



Call points (x, y) with $a \leq x \leq b$ and $c \leq y \leq d$,

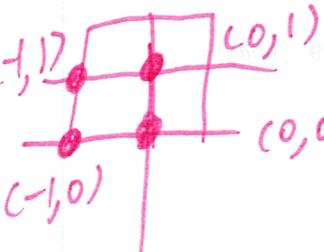
we cut it into $m \times n$ sub rectangles.



We then pick sample points $x_{i,j}$ in the i,j -th subrectangle (upper left, center, etc.)

$$\text{Then } S_{n,m} = \sum_{i=1}^n \sum_{j=1}^m \left(\frac{b-a}{n} \times \frac{d-c}{m} \right) f(x_i, y_j)$$

Ex: $f(x,y) = x^2 + y^2$, $R = [-1, 1] \times [0, 2]$, $n=2, m=2$
with bottom left sample points.

Sol: 

$$S_{2,2} = \left(\frac{1-(-1)}{2} \times \frac{2-0}{2} \right) (f(-1,0) + f(-1,1) + f(0,0) + f(0,1))$$

$$= 1 \cdot 1 (1 + 2 + 0 + 1) = 4.$$

Then the integral is the limit of these Riemann sums:

$$\iint_R f(x,y) dA := \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \frac{b-a}{n} \times \frac{d-c}{m} \sum_{i=1}^n \sum_{j=1}^m f(x_i, y_j)$$

If $f(x,y) \geq 0$ over R , then this integral represents the volume of the region below the graph of f and above R . If f is not always positive, then this is a signed volume like the regular Calc I integral is signed area.

We say f is integrable over R if $\iint_R f(x,y) dA$ exists.

Ex: $f(x,y) = 1$ if $x+y$ is rational, 0 otherwise
is not integrable over any R with positive area.

We don't use limits to compute integrals, just like in Calc I.

An iterated integral is of the form

$$\int_c^d \left(\int_a^b f(x,y) dy \right) dx \quad \text{or} \quad \int_a^b \left(\int_c^d f(x,y) dx \right) dy$$

The idea is the same as for partial derivatives! The variable not being integrated is treated as a constant, and we integrate as in the normal one-variable sense twice.

Ex: $\int_0^3 \left(\int_4^8 ye^{xy} dx \right) dy$

Sol: we treat y as a constant. For example, $y=5$, we would integrate $\int e^{5x} dx$ as $e^{5x} + C$, and it is the same with y !

$$\int_4^8 ye^{xy} dx = e^{xy} \Big|_4^8 = e^{8y} - e^{4y}. \text{ This leaves us with}$$

$$\int_0^3 e^{8y} - e^{4y} dy = \frac{e^{8y}}{8} - \frac{e^{4y}}{4} \Big|_0^3 = \frac{e^{24}}{8} - \frac{e^{12}}{4} - \frac{1}{8} + \frac{1}{4}.$$

Fubini's Theorem: If f is continuous on all but finitely many curves of R , then

$$\iint_R f(x,y) dA = \int_c^d \int_a^b f(x,y) dy dx = \int_a^b \int_c^d f(x,y) dx dy$$

Ex: Let $R = [-3, 2] \times [0, \pi/2]$ and let

$f(x, y) = x \cos(yx)$, compute $\iint_R f(x, y) dA$.

Sol: By Fubini's Theorem,

$$\iint_R f(x, y) dA = \int_{-3}^2 \int_0^{\pi/2} x \cos(yx) dy dx$$

and

$$\iint_R f(x, y) dA = \int_0^{\pi/2} \int_{-3}^2 x \cos(yx) dx dy.$$

One of these is much easier to integrate, so we can use that one.

$$\int_{-3}^2 \int_0^{\pi/2} x \cos(yx) dy dx = \int_{-3}^2 x \sin(yx) \Big|_0^{\pi/2} dx =$$

$$\int_{-3}^2 x \sin\left(\frac{\pi x}{2}\right) dx =$$

$$\int_{-3}^2 x \sin\left(\frac{\pi x}{2}\right) dx =$$

$$-\frac{2}{\pi} \cos\left(\frac{\pi x}{2}\right) \Big|_{-3}^2 =$$

$$-\frac{2}{\pi} \cos(\pi) - \left(-\frac{2}{\pi} \cos\left(-\frac{3\pi}{2}\right)\right)$$

$$= \frac{4}{\pi}.$$