

Triple Integrals

There is nothing stopping us from integrating in higher dimensions. The same Riemann sum process we used in 2D works in 3D. That is, we integrate $f(x, y, z)$ over a 3D box $R = [a, b] \times [c, d] \times [e, f]$. 1D integrals represent area, 2D-integrals represent volume, and we think of 3D integrals as representing mass.

Fubini's Theorem V3

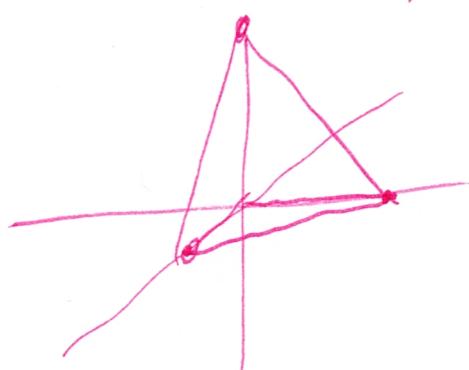
If $D = \{(x, y)\}$: There is some z with (x, y, z) in E^3 (i.e., the "shadow" of E in the xy -plane), and $E = \{(x, y, z) : (x, y) \in D \text{ and } u_1(x, y) \leq z \leq u_2(x, y)\}$, then

$$\iiint_E f(x, y, z) dV = \iint_D \left(\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) dz \right) dA$$

The order of the variables can change. For example, the region D could be in the yz -plane, with x -values being bounded by functions of y and z . Or any combination thereof.

Ex: Let E be the tetrahedron with vertices $(0, 0, 0)$, $(0, 0, 4)$, $(2, 0, 0)$, and $(0, 1, 0)$. Calculate $\iiint_E 3x dV$.

Sol: We need to determine the bounds for E. To do that, we need the plane equation for the shaded face.



The plane contains $\langle -2, 1, 0 \rangle$ and $\langle -2, 0, 4 \rangle$, so its normal vector is $\langle -2, 0, 4 \rangle \times \langle -2, 1, 0 \rangle = \langle -4, -8, 2 \rangle$.

We'll use $\langle 4, 8, 2 \rangle$ since this is a normal vector, thus we

also have $4x + 8y + 2z = d$, which when we plug in $(0, 1, 0)$ yields $4 \cdot 0 + 8 \cdot 1 + 2 \cdot 0 = d$, so $d = 8$. Thus the plane equation is $4x + 8y + 2z = 8$.

To find our z bounds, we rewrite this in terms of z: $z = 4 - 2x - 4y$.

To find our y bounds, we look at the triangular face in the xy-plane.

The lower bound is 0, and the upper bound is given by the line where the plane intersects the xy-plane, i.e. $4x + 8y = 8$, or $y = 1 - \frac{x}{2}$. The x bounds are then from 0 to 2.

Thus, we are finally ready to compute the integral.

$$\begin{aligned}
 & \int_0^2 \int_0^{1-\frac{x}{2}} \int_0^{4-2x-4y} 3x \, dz \, dy \, dx = \int_0^2 \int_0^{1-\frac{x}{2}} 3x z \Big|_0^{4-2x-4y} \, dy \, dx \\
 &= \int_0^2 \int_0^{1-\frac{x}{2}} 12x - 6x^2 - 12xy \, dy \, dx = \int_0^2 (12xy - 6x^2y - 6xy^2) \Big|_0^{1-\frac{x}{2}} \, dx \\
 &= \int_0^2 12x - 6x^2 - 6x^2 + 3x^3 - 6x + 6x^2 - \frac{3}{2}x^3 \, dx = \\
 & \quad \int_0^2 \frac{3}{2}x^3 - 6x^2 + 6x \, dx = (\frac{3}{8}x^4 - 2x^3 + 3x^2) \Big|_0^2 = 6 - 16 + 12 = 2.
 \end{aligned}$$

If we want to swap the order of integration, we just need to rearrange the formulas appropriately.

Ex: Calculate the volume of the above tetrahedron.

Sol: To find volume, we integrate $f(x,y,z)=1$, we shall use the order $dy \, dx \, dz$. Then $y = 1 - \frac{x}{2} - \frac{z}{4}$ is our upper y bound, The triangle in the xz plane is given by $y=0$, $4x+2z=8$. The upper x bound is then $2 - \frac{3}{2}z$, and the upper z bound is 4. Thus the volume is given by

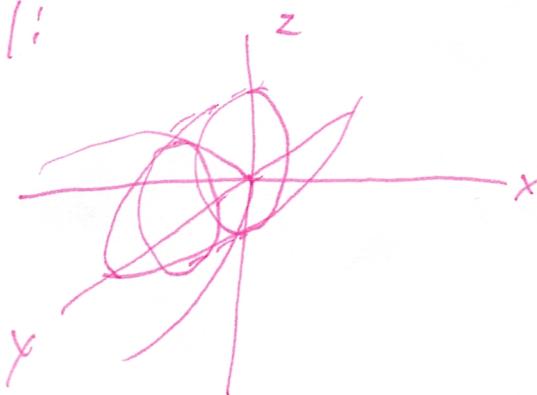
$$\begin{aligned}
 & \int_0^4 \int_0^{2-\frac{3}{2}z} \int_0^{1-\frac{x}{2}-\frac{z}{4}} 1 \, dy \, dx \, dz = \int_0^4 \int_0^{2-\frac{3}{2}z} 1 - \frac{x}{2} - \frac{z}{4} \, dx \, dz = \\
 & \quad \int_0^4 \left(x - \frac{x^2}{4} - \frac{zx}{4} \right) \Big|_0^{2-\frac{3}{2}z} \, dz = \int_0^4 2 - \frac{z}{2} - \frac{(2-\frac{3}{2}z)^2}{4} + \frac{z^2}{8} - \frac{z}{2} \, dz =
 \end{aligned}$$

$$\int_0^4 2 - z + \frac{z^2}{8} - \frac{(2 - \frac{z}{2})^2}{4} dz = (2z - \frac{z^2}{2} + \frac{z^3}{24} + \frac{(2 - \frac{z}{2})^3}{6}) \Big|_0^4$$

$$= 8 - 8 + \frac{64}{24} - \frac{8}{6} = \frac{64}{24} - \frac{32}{24} = \frac{32}{24} = \frac{4}{3}$$

Ex: Calculate $\iiint_E z dV$, where E is the region bounded by $y = x^2 + z^2$ and $y = 2 - x^2 - z^2$.

Sol:



As y has nice bounds in terms of x and z , it will be our inner variable.

The outer bounds will then be determined by the shadow on the xz -plane,

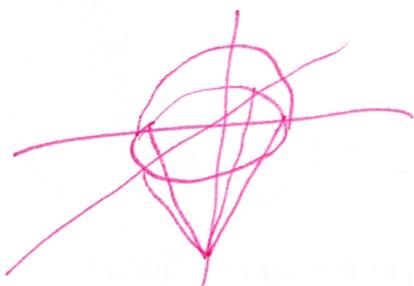
This is at where the graphs meet, i.e., $x^2 + z^2 = 2 - x^2 - z^2$, $x^2 + z^2 = 1$. That is, the unit circle. Therefore, we have

$$\begin{aligned} & \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{x^2+z^2}^{2-x^2-z^2} z dy dz dx = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} z(2-2x^2-2z^2) dz dx \\ &= \int_{-1}^1 \left(\frac{1-x^2-z^2}{2} \right)^2 \Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx = \int_{-1}^1 0 dx = 0. \end{aligned}$$

For more complicated regions, we can integrate by splitting them up into simpler regions just like for two variable integrals.

Ex: Find the value of $\iiint_E 3 \, dV$, where E is the region bounded by $z = \sqrt{1-x^2-y^2}$ and $z = \sqrt{x^2+y^2} - 1$.

Sol:



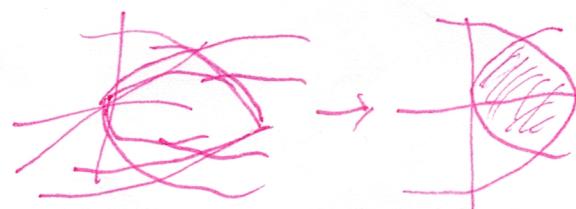
This shape is a half sphere cut off at the top or a cone with radius and height 1. Then

$$\iiint_E 3 \, dV = 3 \iiint_E 1 \, dV = 3 \iiint_S 1 \, dV + 3 \iiint_C 1 \, dV,$$

The latter two integrals are the volumes, $\frac{2}{3}\pi$ and $\frac{\pi}{3}$ respectively, so the total value is 3π .

Tips for drawing regions:

- Geogebra (homework + webwork)
- Project into 2D! Ex:



- Find an axis of symmetry, i.e., $z = x^2 + y^2$ is a parabola rotated around the z -axis.