

Ex: Integrate  $xy$  over the region  $R$   
 defined via  $y \leq z \leq x$ ,  $0 \leq y \leq x^2$ ,  $0 \leq x \leq 3$

Sol: Note that for  $x > 1$ ,  $x^2 > x$ , so  
 $y$  cannot range up to  $x^2$  because  
 $y \leq z \leq x$  could not be satisfied.

Therefore, we need to split this integral.

$$\iiint_E xy \, dv = \int_0^1 \int_0^{x^2} \int_y^x xy \, dz \, dy \, dx + \int_1^3 \int_0^{x^2} \int_y^x xy \, dz \, dy \, dx$$

$$= \int_0^1 \int_0^{x^2} x^2 - y^2 \, dy \, dx + \int_1^3 \int_0^{x^2} x^2 - y^2 \, dy \, dx$$

$$= \int_0^1 (xy - \frac{y^3}{3}) \Big|_0^{x^2} \, dx + \int_1^3 (xy - \frac{y^3}{3}) \Big|_0^x \, dx$$

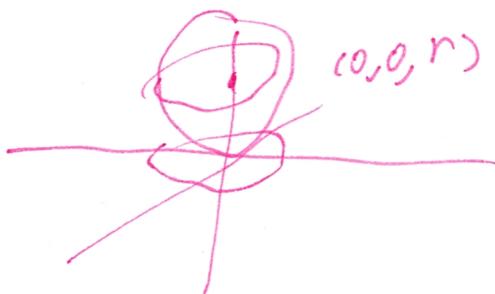
$$= \int_0^1 x^4 - \frac{x^6}{3} \, dx + \int_1^3 x^3 - \frac{x^3}{3} \, dx = \int_0^1 x^4 - \frac{8}{3} \, dx + \int_1^3 x^3 \, dx$$

$$= \cancel{\int_0^1 (x^5 - \frac{x^7}{21}) \Big|_0^1} + (\frac{x^4}{4}) \Big|_1^3 =$$

$$\frac{1}{5} - \frac{1}{21} + \frac{27}{2} - \frac{1}{6}$$

Ex: write down an integral that  
 calculates the mass of a sphere  
 centered at  $(0, 0, r)$  with radius  $r$   
 and density  $f(x, y, z) = x^2y + z$ .

Sol:



The top half of a sphere at the origin with radius  $r$  is  $\sqrt{r^2 - x^2 - y^2}$ . The bottom half is  $-\sqrt{r^2 - x^2 - y^2}$ .

At both have been shifted up by  $\delta$ . They are  $r + \sqrt{r^2 - x^2 - y^2}$  and  $r - \sqrt{r^2 - x^2 - y^2}$  for E. The shadow in the xy-plane is the circle of radius  $r$ . Therefore the mass is given by

$$\iiint_E f(x, y, z) dV = \int_{-r}^r \int_{-\sqrt{r^2 - x^2}}^{\sqrt{r^2 - x^2}} \int_{r - \sqrt{r^2 - x^2 - y^2}}^{r + \sqrt{r^2 - x^2 - y^2}} x^2 y + z dz dy dx$$

Actually integrating this will be quite challenging. In general, circles and spheres will be harder to integrate because copious square roots of squares. Next week, we will develop techniques to make this easier.

Tips for visualizing 3D regions:

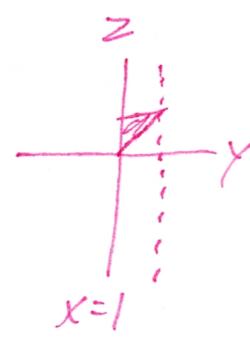
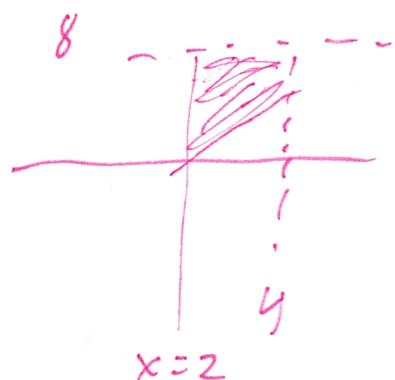
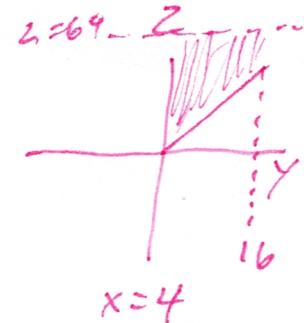
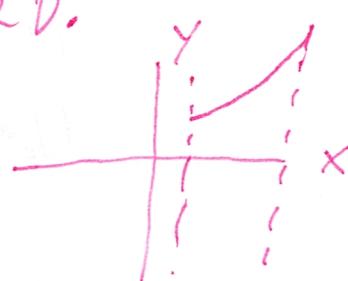
- Geogebra (homework + Webwork)

- Project into 2D:

$$\text{Ex: } 1 \leq x \leq 4$$

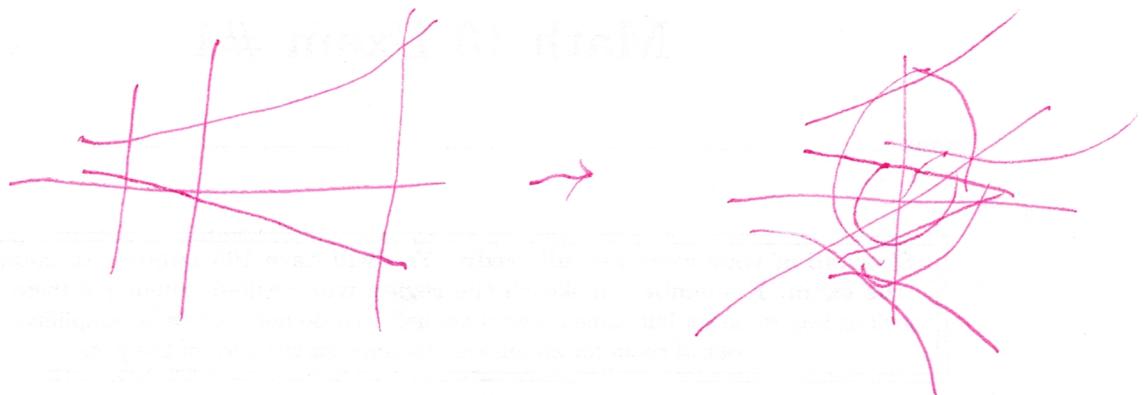
$$0 \leq y \leq x^2$$

$$y \leq z \leq x^3$$

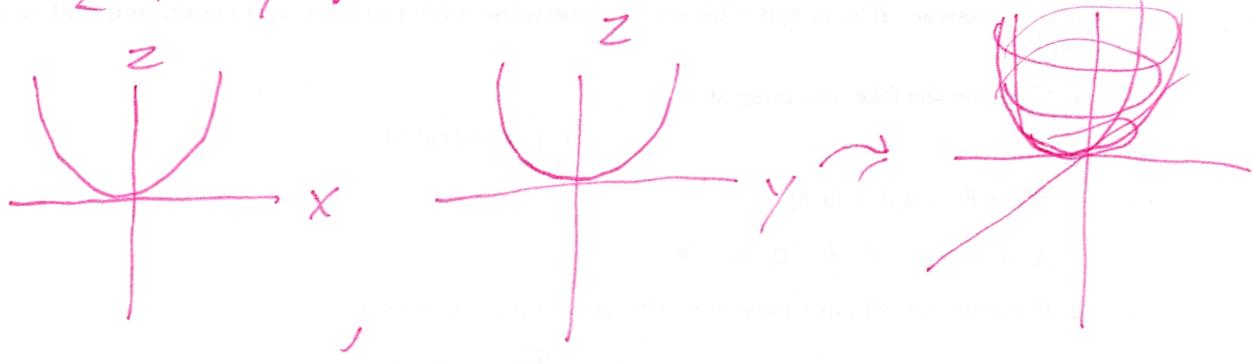


Find an axis of symmetry:

- Washer problems from Calc I:



$$- z = x^2 + y^2,$$



These can also be used to compute easily side step having to compute an integral.

Ex: Let E be bounded by  $z = x^2 + y^2$  and  $z = 3$ . Compute  $\iiint_E y \, dV$ .

Sol: Notice that the shape is symmetric along the y-axis and the function y is odd: for every contribution of y, there is a contribution of -y. So  $\iiint_E y \, dV = 0$ .