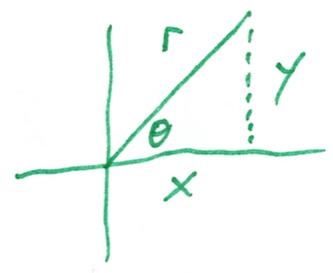


Polar Coordinates and Cylindrical Coordinates

If we want to write a treasure map, there are two ways to write instructions.
 I.e.: - "Take five steps east, three steps north"
 - "Face north-north east, take seven steps."

Can Cartesian coordinates correspond to the former, Polar coordinates correspond to the latter.



θ represents the angle counter-clockwise from the positive x-axis. It ranges from 0 to 2π . r represents the distance from the origin to travel.

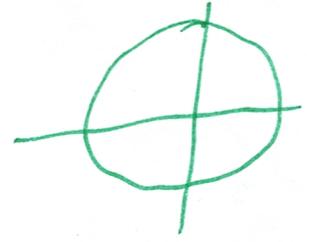
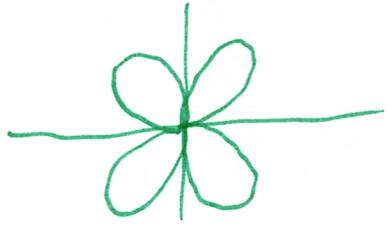
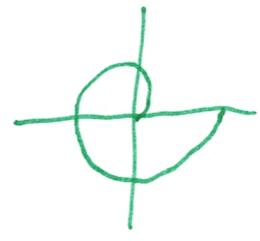
Given (x, y) , $r = \sqrt{x^2 + y^2}$ and $\tan \theta = \frac{y}{x}$,
 $0 \leq \theta < 2\pi$, $\theta = \begin{cases} \arctan(\frac{y}{x}) & x > 0 \\ \pi + \arctan(\frac{y}{x}) & x < 0 \end{cases}$.

Polar coordinates define functions just like Cartesian coordinates.

$r = \theta$

$r = 3 \sin 2\theta$

$r = 4$

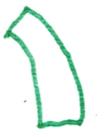


Some regions are much easier to describe in Polar coordinates, making them easier to integrate over as well.

Just like with Cartesian coordinates, if $f(r, \theta)$ is defined over D , then we have

$$\iint_D f(r, \theta) dA$$

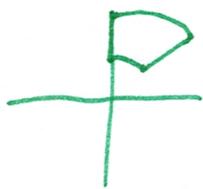
However dA is not equal to $dr d\theta$:
 When computing our Riemann sums, the small regions are of the shape



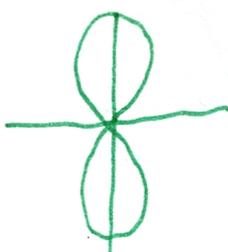
This is a bent rectangle with width dr , however, the height is $r d\theta$, so for polar coordinates, $dA = r dr d\theta$.

Ex: $\pi/4 \leq \theta \leq \pi/2$, $1 \leq r \leq 3$. Calculate $\iint_D \frac{1}{r} dA$.

Sol: $\iint_D \frac{1}{r} dA = \int_{\pi/4}^{\pi/2} \int_1^3 \frac{1}{r} \cdot r dr d\theta = \int_{\pi/4}^{\pi/2} \int_1^3 1 dr d\theta$
 $= 2 \int_{\pi/4}^{\pi/2} 1 d\theta = \pi/2$.



Ex: Integrate x over the region $0 \leq r \leq 3 \sin \theta$.



$$\int_0^{2\pi} \int_0^{3 \sin \theta} r^2 \cos \theta dr d\theta = \int_0^{2\pi} \left. \frac{r^3}{3} \cos \theta \right|_0^{3 \sin \theta} d\theta =$$

$$\int_0^{2\pi} 9 \sin^3 \theta \cos \theta d\theta = \left. \frac{9 \sin^4 \theta}{4} \right|_0^{2\pi} = 0.$$

Switching the order of integration works the same as it does with Cartesian coordinates, but is rarely done.

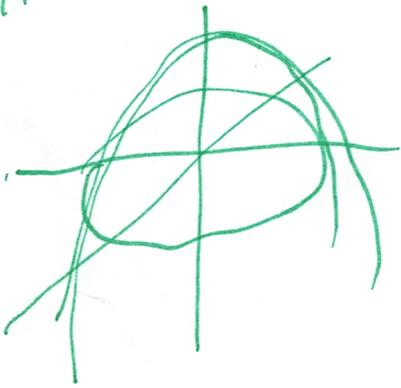
Recall that $\iiint_E f(x, y, z) dV = \iint_D \int_{u(x, y)}^{v(x, y)} f(x, y, z) dz dA$, there is nothing stopping us from doing the double integral in polar

coordinates rather than Cartesian, This gives us cylindrical coordinates (r, θ, z) , where r, θ act in the xy -plane according to polar coordinates and z acts along the z -axis according to Cartesian coordinates.

In cylindrical coordinates, $dV = dz(dA) = r dz dr d\theta$.

Ex: Compute the mass of the solid region E , where E is the region bounded above the xy -plane by $z = 4 - x^2 - y^2$ and the density is given by $f(x, y, z) = z^2$.

Sol:



The shadow in the xy -plane is the circle of radius 2. So this

integral is given by

$$\int_0^{2\pi} \int_0^2 \int_0^{4-r^2} z^2 r dz dr d\theta =$$

$$\int_0^{2\pi} \int_0^2 \frac{(4-r^2)^3}{3} r dr d\theta =$$

$$\int_0^{2\pi} -\frac{(4-r^2)^4}{24} \Big|_0^2 d\theta =$$

$$\int_0^{2\pi} \frac{32}{3} d\theta = \frac{64}{3} \pi$$