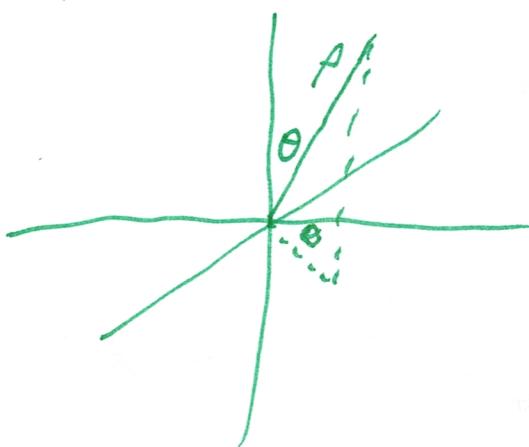


Spherical coordinates

Polar coordinates are an alternate way to navigate 2D space. Spherical coordinates are an alternate way to navigate 3D space.



θ still represents our angle from the positive x-axis in the xy-plane and ranges from 0 to 2π .

ϕ represents the angle from the positive z-axis and ranges from 0 to π . (It does not range from 0 to 2π because that would double count certain points.)

ρ is the distance from the origin.

$$x = \rho \sin \phi \cos \theta$$

$$y = \rho \sin \phi \sin \theta$$

$$z = \rho \cos \phi$$

$$\sqrt{x^2 + y^2} = r = \rho \sin \phi$$

$$x^2 + y^2 + z^2 = \rho^2$$

$$r = \rho \sin \phi$$

$$\theta = \theta$$

$$z = \rho \cos \phi$$

$$\therefore \rho = \sqrt{x^2 + y^2 + z^2}$$

$$\begin{cases} \theta = \arctan(\frac{y}{x}), x > 0 \\ \arctan(\frac{y}{x}) + \pi, x < 0 \end{cases}$$

$$\phi = \arccos(z)$$

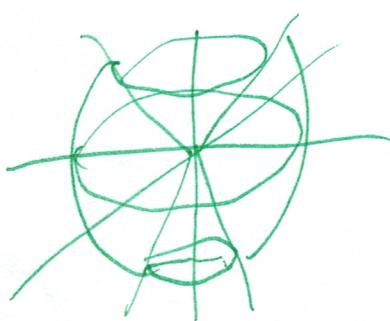
When integrating with respect to spherical coordinates, the basic shapes of our Riemann sum are of the form

$$\text{Then } dV = (\rho \sin\phi d\theta d\phi) d\rho = \rho^2 \sin\phi d\phi d\theta d\rho$$



Ex: Find the volume of the part of the sphere of radius 2 between the cones $\phi = \pi/3$ and $\phi = 2\pi/3$.

Sol:



$$\begin{aligned} V &= \iiint_E 1 \, dV = \int_0^{2\pi} \int_0^2 \int_{\pi/3}^{2\pi/3} \rho^2 \sin\phi \, d\phi \, d\rho \, d\theta \\ &= \int_0^{2\pi} \int_0^2 \rho^2 - \cos\phi \Big|_{\pi/3}^{2\pi/3} \, d\rho \, d\theta \\ &= \int_0^{2\pi} \int_0^2 \rho^2 \, d\rho \, d\theta = \int_0^{2\pi} \frac{\rho^3}{3} \Big|_0^2 = \int_0^{2\pi} \frac{8\pi}{3} \, d\theta \\ &= \frac{16\pi}{3}. \end{aligned}$$

Ex: Evaluate $\int_0^3 \int_0^{\sqrt{9-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} (x^2+y^2+z^2) \, dz \, dy \, dx$ by transforming into spherical coordinates.

Sol: $x^2+y^2+z^2=\rho^2$, and $dV=\rho^2 \sin\phi \, d\phi \, d\theta \, d\rho$, so the function we are integrating is $\rho^4 \sin\phi$. $z=\sqrt{x^2+y^2}$ is $z=r$, which in spherical coordinates is $\rho \cos\phi = \rho \sin\phi$, or $\cos\phi = \sin\phi$. This happens for $\phi = \pi/4$, so $\phi \leq \pi/4$. $z=\sqrt{18-x^2-y^2}$ implies $z^2=18-x^2-y^2$ or $\rho^2=18$. Thus $\rho \leq 3\sqrt{2}$.

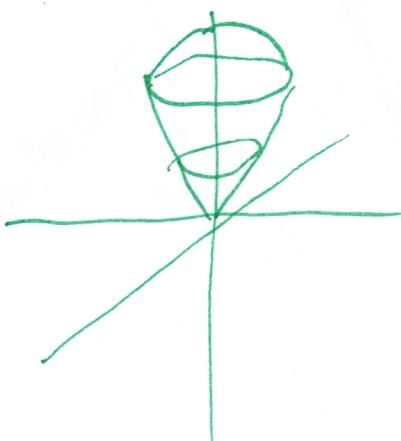
Finally, note that $S_0^3 S_0^{3\pi/4}$ represents the quarter-circle of radius 3 in the first quadrant. Thus $0 \leq \theta \leq \pi/2$ and $0 \leq \rho \sin \phi \leq 3$. $\sin \phi$ is maximized as $\frac{\sqrt{2}}{2}$, so $\sqrt{2}\rho \leq 3$. This simplifies to $\rho \leq \frac{6}{\sqrt{2}} = \cancel{\frac{6}{\sqrt{2}}} \frac{2}{3\sqrt{2}}$.

Thus our integral is

$$\begin{aligned} & \int_0^{\pi/2} \int_0^{3\sqrt{2}} \int_0^{\pi/4} \rho^4 \sin \phi d\phi d\rho d\theta = \int_0^{\pi/2} \int_0^{3\sqrt{2}} -\cos \phi \rho^4 l_0^{1/4} d\rho d\theta \\ &= \int_0^{\pi/2} \int_0^{3\sqrt{2}} \left(\frac{2-\sqrt{2}}{2}\right) \rho^4 d\rho d\theta = \int_0^{\pi/2} \left(\frac{2-\sqrt{2}}{2}\right) \rho^5 l_0^{1/4} d\theta \\ &= \int_0^{\pi/2} \frac{3^5 \cdot 2 \cdot \sqrt{2}}{5} (2-\sqrt{2}) d\theta = \frac{3^7 \cdot \sqrt{2} \cdot (2-\sqrt{2}) \pi}{5}. \end{aligned}$$

Ex: Find the mass of the solid E bounded above by $\sqrt{9-x^2-y^2}$ and below by $\sqrt{3x^2+3y^2}$ with density $\delta(x, y, z) = \sqrt{x^2+y^2+z^2}$.

Sol: In spherical coordinates, this is



$$\begin{aligned} & \delta(\rho, \theta, \phi) = \rho, \quad (\cot \phi = \sqrt{3} \text{ occurs}) \\ & 2 = \sqrt{9-x^2-y^2}, \quad \text{when } \phi = \pi/6. \\ & = x^2+y^2=9, \quad \text{therefore, mass is} \\ & = \rho = 3, \quad \text{given by} \\ & z = \sqrt{3x^2+3y^2}, \quad \int_0^{2\pi} \int_0^3 \int_0^{\pi/6} \rho^3 \sin \phi d\phi d\rho d\theta \\ & = z^2 = 3x^2+3y^2, \quad = \int_0^{2\pi} \int_0^3 -\cos \phi \rho^3 l_0^{1/6} d\rho d\theta \\ & = z^2 = 3r^2, \quad = \int_0^{2\pi} \int_0^3 \left(\frac{2-\sqrt{3}}{2}\right) \rho^3 d\rho d\theta \\ & = z = \sqrt{3} r, \quad = \int_0^{2\pi} \left(\frac{2-\sqrt{3}}{2}\right) \frac{\rho^4}{4} l_0^{1/3} d\theta \\ & = \rho \cos \phi = \sqrt{3} \rho \sin \phi, \quad = \int_0^{2\pi} \left(\frac{2-\sqrt{3}}{2}\right) \frac{81}{4} d\theta = \\ & \cot \phi = \sqrt{3}. \end{aligned}$$