

Finally, note that  $\int_0^{\pi/2} \int_0^{3\sqrt{2}} r^3 \sin \phi dr d\theta$  represents the quarter-circle of radius 3 in the first quadrant. Thus  $0 \leq \theta \leq \pi/6$  and  $0 \leq \rho \sin \phi \leq 3$ .  $\sin \phi$  is maximized as  $\frac{\sqrt{2}}{2}$ , so  $\frac{\sqrt{2}\rho}{2} \leq 3$ . This simplifies to  $\rho \leq \frac{6}{\sqrt{2}} = \cancel{\frac{6}{\sqrt{2}}} \frac{2}{3\sqrt{2}}$ .

Thus our integral is

$$\begin{aligned} & \int_0^{\pi/2} \int_0^{3\sqrt{2}} \int_0^{\pi/4} \rho^4 \sin \phi d\phi d\rho d\theta = \int_0^{\pi/2} \int_0^{3\sqrt{2}} -\cos \phi \rho^4 |_0^{\pi/4} d\rho d\theta \\ &= \int_0^{\pi/2} \int_0^{3\sqrt{2}} \left(\frac{2-\sqrt{2}}{2}\right) \rho^4 d\rho d\theta = \int_0^{\pi/2} \left(\frac{2-\sqrt{2}}{2}\right) \rho^5 |_0^{3\sqrt{2}} d\theta \\ &= \int_0^{\pi/2} \frac{3^5 \cdot 2 \cdot \sqrt{2}}{5} (2-\sqrt{2}) d\theta = \frac{3^5 \cdot \sqrt{2} \cdot (2-\sqrt{2})}{5} \pi \end{aligned}$$

Ex: Find the mass of the solid E bounded above by  $\sqrt{9-x^2-y^2}$  and below by  $\sqrt{3x^2+3y^2}$  with density  $\delta(x, y, z) = \sqrt{x^2+y^2+z^2}$ .

Sol: In spherical coordinates this is

$$\delta(\rho, \theta, \phi) = \rho, \quad (\cot \phi = \sqrt{3} \text{ occurs when } \phi = \pi/6)$$

$$2 = \sqrt{9-x^2-y^2}, \quad \text{when } \phi = \pi/6.$$

$$= x^2 + y^2 = 9, \quad \text{therefore, mass is given by}$$

$$Z = \sqrt{3x^2+3y^2}, \quad \int_0^{2\pi} \int_0^3 \int_0^{\pi/6} \rho^3 \sin \phi d\phi d\rho d\theta$$

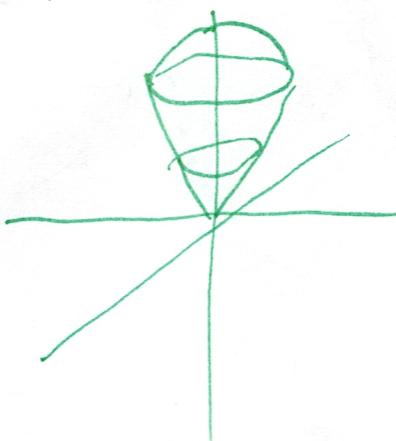
$$= Z^2 = 3x^2 + 3y^2, \quad = \int_0^{2\pi} \int_0^3 -\cos \phi \rho^3 |_0^{\pi/6} d\phi d\theta$$

$$= Z^2 = 3r^2, \quad = \int_0^{2\pi} \int_0^3 \left(\frac{2-\sqrt{3}}{2}\right) \rho^3 d\rho d\theta$$

$$= Z = \sqrt{3}r, \quad = \int_0^{2\pi} \left(\frac{2-\sqrt{3}}{2}\right) \frac{\rho^4}{4} |_0^3 d\theta$$

$$= \rho \cos \phi = \sqrt{3} \rho \sin \phi, \quad = \int_0^{2\pi} \left(\frac{2-\sqrt{3}}{2}\right) \frac{81}{4} d\theta =$$

$$\cot \phi = \sqrt{3}.$$



$$\left(\frac{2-\sqrt{3}}{2}\right)\frac{81}{4} \text{ or } 2^x = (2-\sqrt{3}) \cdot \frac{81}{4} \cdot \pi.$$

We can use triple integrals to compute physical properties of objects. If  $E$  is a 3D region with density function  $\delta(x, y, z)$ , then

$$\text{Mass} = \iiint_E \delta(x, y, z) dV$$

The moment along the  $x$ -axis is given by

$$M_x = M_{yz} = \iiint_E x \delta(x, y, z) dV$$

Similarly for  $y$  and  $z$ ,

$$M_y = M_{xz} = \iiint_E y \delta(x, y, z) dV$$

$$M_z = M_{xy} = \iiint_E z \delta(x, y, z) dV$$

The center of mass of  $E$  is given by

$$C_m = \left( \frac{M_x}{\text{Mass}}, \frac{M_y}{\text{Mass}}, \frac{M_z}{\text{Mass}} \right)$$

The center of mass is the balance point of an object.

Ex: Find the center of mass of a sphere of radius  $R$ , with density  $\delta(x, y, z) = 2$ .

Sol: We know  $C_m = (0, 0, 0)$ , or it should be. We double check this with triple integrals in spherical coordinates.

$$\begin{aligned} \text{Mass} &= \int_0^{2\pi} \int_0^R \int_0^\pi \rho^2 \sin\phi d\phi d\rho d\theta = \int_0^{2\pi} \int_0^R -\cos\phi \rho^2 \Big|_0^\pi d\rho d\theta \\ &= \int_0^{2\pi} \int_0^R 2\rho^2 d\rho d\theta = \int_0^{2\pi} \frac{2\rho^3}{3} \Big|_0^R d\theta = \int_0^{2\pi} \frac{2R^3}{3} d\theta = \frac{4\pi R^3}{3} \end{aligned}$$

$$M_x = \iiint_E x \delta(x, y, z) dV = \int_0^{2\pi} \int_0^R \int_0^\pi \rho^3 \sin^2 \phi \cos \theta d\rho d\phi d\theta$$

$$= \int_0^{2\pi} \int_0^R \rho^3 \cos \theta \left( \frac{\pi}{2} - \frac{\sin(2\theta)}{4} \right) \Big|_0^\pi d\rho d\theta =$$

$$\int_0^{2\pi} \int_0^R \frac{\pi}{2} \rho^3 \cos^2 \theta d\rho d\theta = \int_0^{2\pi} \frac{\pi R^4}{8} \cos^2 \theta d\theta =$$

$$\frac{\pi R^4}{8} \sin \theta \Big|_0^{2\pi} = 0.$$

$$M_y = \iiint_E y \delta(x, y, z) dV = \int_0^{2\pi} \int_0^R \int_0^\pi \rho^3 \sin^2 \phi \sin \theta d\rho d\phi d\theta$$

$$= \int_0^{2\pi} \int_0^R \frac{\pi R^4}{8} \sin^2 \theta d\theta = \frac{\pi R^4}{8} (-\cos \theta) \Big|_0^{2\pi} = 0$$

$$M_z = \iiint_E z \delta(x, y, z) dV = \int_0^{2\pi} \int_0^R \int_0^\pi \rho^3 \sin^2 \phi \cos \theta d\rho d\phi d\theta =$$

$$\int_0^{2\pi} \int_0^R \frac{\rho^3 \sin^2 \phi}{2} \Big|_0^\pi d\rho d\theta = \int_0^{2\pi} \int_0^R 0 d\rho d\theta = 0$$

Thus  $\mathbf{G}_M = \left( \frac{0}{\frac{4\pi R^3}{3}}, \frac{0}{\frac{4\pi R^3}{3}}, \frac{0}{\frac{4\pi R^3}{3}} \right) = (0, 0, 0)$  as expected.

We could have already guessed this from symmetry: IF  $E$  is symmetric along the  $x/y/z$ -axis, and  $\delta(x, y, z)$  is even in  $x/y/z$ , then  $x\delta(x, y, z)/y\delta(x, y, z)/z\delta(x, y, z)$  is odd in  $x/y/z$ . Thus  $\iiint_E x\delta(x, y, z) dV = 0$  /  $\iiint_E y\delta(x, y, z) dV = 0$  /  $\iiint_E z\delta(x, y, z) dV = 0$ .

Ex: Find the center of mass of  $E$ , the solid bounded by  $z = 4 - x^2 - y^2$  and  $z = 0$ , with  $\delta(x, y, z) = 1$ .

Sol: We integrate using cylindrical coordinates.  $M_x = M_y = 0$  by symmetry.

$$\text{Mass} = \int_0^{2\pi} \int_0^2 \int_0^{4-r^2} r^2 z dz dr d\theta = \int_0^{2\pi} \int_0^2 \frac{2}{3} r^3 \Big|_0^{4-r^2} dr d\theta =$$

$$S_0^{2\pi} S_0^2 \frac{(4-r^2)^3}{3} r dr d\theta = S_0^{2\pi} \frac{(4-r^2)^4}{24} r^2 dr d\theta = S_0^{2\pi} \frac{32}{3} r^2 dr d\theta = \frac{64\pi}{3}$$

$$M_2 = S_0^{2\pi} S_0^2 \int_0^{4-r^2} z^3 r dz dr d\theta = S_0^{2\pi} S_0^2 \frac{z^4}{4} r \Big|_0^{4-r^2} dr d\theta =$$

$$S_0^{2\pi} S_0^2 \frac{(4-r^2)^4}{4} r dr d\theta = S_0^{2\pi} \frac{(4-r^2)^5}{40} r^2 dr d\theta =$$

$$S_0^{2\pi} \frac{128}{5} r dr d\theta = \frac{256\pi}{5}$$

Therefore  $(M) = (0, 0, \frac{256\pi}{5}) = (0, 0, \frac{12}{5})$ ,

We can use spherical coordinates with different axes than usual.

Eg: Let  $E$  be the region inside the sphere of radius 3 bounded between  $x = \sqrt{y^2+z^2}$  and  $x = -\sqrt{y^2+z^2}$ . Find the volume of  $E$ .

Sol:



Let  $\theta$  measure the angle from the positive  $y$ -axis in the  $yz$ -plane, and let  $\phi$  measure the angle from the positive  $x$ -axis. Then

$$\text{Volume}(E) = S_0^{2\pi} S_0^3 \int_{\pi/4}^{3\pi/4} \rho^2 \sin\theta d\phi d\rho d\theta =$$

$$S_0^{2\pi} S_0^3 \left[ -\cos\theta \rho^2 \right]_{\pi/4}^{3\pi/4} d\phi d\theta =$$

$$S_0^{2\pi} S_0^3 \sqrt{2} \rho^2 d\phi d\theta = S_0^{2\pi} \sqrt{2} \frac{\rho^3}{3} \Big|_0^3 d\phi d\theta =$$

$$S_0^{2\pi} 9\sqrt{2} d\phi d\theta = 18\sqrt{2}\pi.$$

(Note this is the same as finding the volume of the shape rotated ~~about~~ to the  $z$ -axis.)