

## Change of Variables

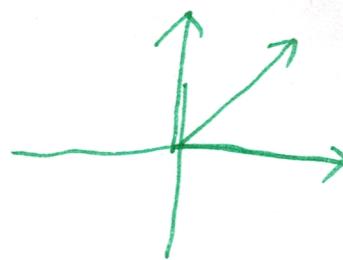
Change of variables is like u-substitution for multiple integrals.

A function  $G: X \rightarrow Y$  (from  $X$  to  $Y$ ) is sometimes called a map.  $X$  is called the domain and  $Y$  is called the codomain. Given A a subset of  $X$ ,  $G(A)$ , the image of  $A$ , is  $\{y \in Y : \text{There is } x \in A \text{ with } G(x)=y\}$ . The image of  $X$ ,  $G(X)$ , is called the range of  $G$ . I.e., it is the collection of all  $y$  in  $Y$  which are of the form  $G(x)$  for  $x$  in  $X$ .

Ex: Let  $G: [0, \infty) \times [0, 2\pi) \rightarrow \mathbb{R}^2$  be defined by  $G(r, \theta) = (r \cos \theta, r \sin \theta)$ .

- Domain:  $[0, \infty) \times [0, 2\pi)$
- Codomain:  $\mathbb{R}^2$
- Image:  $\mathbb{R}^2$
- $G(r, 0) = (r, 0)$  (The non-negative  $x$ -axis)
- $G(r, \pi/2) = (0, r)$  (The non-negative  $y$ -axis)

To find the image of a line or curve under a map, we plug in the equation/relationship defining it.



We use  $u, v$  as variables in the domain and  $x, y$  in the codomain.

$G$  is injective if different points in the domain are all mapped to different points in the range. Algebraically, whenever  $G(u, v) = G(x, y)$ , we have  $u=x$  and  $v=y$ .

Ex: The polar coordinate map  $P$  is not injective on  $[0, \infty) \times [0, 2\pi)$ :  $P(0, \pi) = P(0, 0)$ , but  $0 \neq \pi$ . It is injective on  $(0, \infty) \times [0, 2\pi)$ .

Ex: The map  $G(u, v) = (e^u, e^{u+v})$  is injective on  $\mathbb{R}^2$ . To prove this, we use algebra. Assume  $G(u, v) = G(x, y)$ , i.e.,  $(e^u, e^{u+v}) = (e^x, e^{x+y})$ . This gives us the system of equations:

$$e^u = e^x \quad \text{and} \quad e^{u+v} = e^{x+y}$$

$$\ln e^u = \ln e^x \quad e^{u+v} = e^{u+y}$$

$$u = x \quad \ln e^{u+v} = \ln e^{u+y}$$

$$u+v = u+y$$

$$v = y.$$

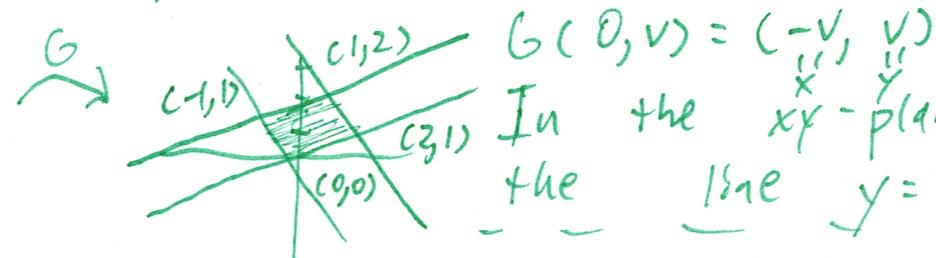
Thus  $G$  is injective, as  $G(u, v) = G(x, y)$  implies  $u=x$  and  $v=y$ .

A transformation  $G$  is linear if it is of the form  $G(u, v) = (Au + Bv, Cu + Dv)$ . It is affine if it is of the form  $G(u, v) = (Auu + Bu + n, Cuv + Du + m)$  for numbers  $n$  and  $m$ .

I.e., an affine map is a linear map plus a constant shift. Linear and affine maps take lines to lines, and thus parallelograms to parallelograms.

Ex: Let  $G(u, v) = (2u - v, u + v)$ . Find the image of  $[0, 1] \times [0, 1]$ .  $[0, 1] \times [0, 1]$  is in the  $uv$ -plane, find its corresponding shape in the  $xy$ -plane.

Sol:



$$G(0, v) = (-v, v)$$

In the  $xy$ -plane, this is the line  $y = -x$ .

$$\begin{aligned} G(u, 0) &= (2u, u) & G(u, 1) &= (2u+1, u+1) & G(1, v) &= (1-v, 1+v) \\ u = \frac{x}{2}, \quad y &= \frac{x}{2}, & x = 2u-1, \quad y = \frac{x}{2} + \frac{1}{2}, & & 2-v = x, \quad y &= 1+v \\ & & \frac{x+1}{2} = u, & & x+v = 2, & y = 1+2-x \\ & & y = u+1, & & x = 2-u, & y = 1+2-x \\ & & y = \frac{x+1}{2} + 1, & & v = 2-x, & x = 3-x \end{aligned}$$

This trick of looking at lines can only be done with linear and affine maps.

Let  $G$  be a map from the  $uv$ -plane into the  $xy$ -plane. The Jacobian matrix for  $G$  is  $\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$ . Then

$\text{Jac}(G)$  is the determinant of the Jacobian matrix,  $\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}$ .

Ex: For a linear map  $G(u, v) = (Au + Bv, Cu + Dv)$ , the Jacobian matrix is  $\begin{pmatrix} A & B \\ C & D \end{pmatrix} = AD - BC$ .