

Change of Variables Formula:

If D is in the uv -plane and G from the uv -plane to the xy -plane is continuous and injective on the interior of D , then

$$\iint_{G(D)} f(x,y) dA = \iint_D f(x(u,v), y(u,v)) |Jac(G)| dA.$$

Ex: Let R be the parallelogram with vertices $(0,0)$, $(-1,1)$, $(2,1)$, and $(1,2)$. Calculate $\iint_R x^2 dA$.

Sol: From the previous example, we know that under the map $G(u,v) = (2u-v, u+v)$,

$G([0,1] \times [0,1]) = R$. Therefore, by the change of variables formula,

$$\iint_R f(x,y) dA = \iint_R x^2 dA = \iint_{G([0,1] \times [0,1])} x^2 dA =$$

$$\iint_{[0,1] \times [0,1]} x^2 |Jac(G)| dA = \iint_{[0,1] \times [0,1]} (2u-v)^2 \cdot 3 \cdot du dv =$$

$$\int_0^1 \int_0^1 3(2u-v)^2 du dv = \int_0^1 \frac{1}{2} (2u-v)^3 \Big|_0^1 dv =$$

$$\frac{1}{2} \int_0^1 (2-v)^3 + v^3 dv = \frac{1}{2} \left(-\frac{(2-v)^4}{4} + \frac{v^4}{4} \right) \Big|_0^1 = \frac{1}{2} \left(-\frac{1}{4} + \frac{1}{4} + 4 - 0 \right) = 2.$$

Note that for $P(r, \theta) = (r \cos \theta, r \sin \theta)$,

$$Jac(G) = \det \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix} = r \cos^2 \theta + r \sin^2 \theta = r.$$

Thus polar coordinates are a specific case of the change of variables formula.

Ex: $G(u, v) = (\frac{u}{v}, \frac{u}{v^2})$, $D = [1, 2] \times [1, 2]$. Compute $\iint_{G(D)} (x^2 + y^2) dA$.

Sol: $|\text{Jac}(G)| = \left| \det \begin{pmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ \frac{1}{v} & u \end{pmatrix} \right| = \frac{u}{v} + \frac{u}{v} = \frac{2u}{v}$.

$x^2 + y^2 = \frac{u^2}{v^2} + u^2 v^2$, $(x^2 + y^2) |\text{Jac}(G)| = \frac{2u^3}{v^3} + 2u^3 v$.

Thus $\iint_{G(D)} (x^2 + y^2) dA = \int_1^2 \int_1^2 \left(\frac{2u^3}{v^3} + 2u^3 v \right) du dv =$

$\int_1^2 \left(\frac{u^4}{2v^3} + \frac{u^4}{2} v \right) \Big|_1^2 dv = \int_1^2 \left(\frac{15}{2v^3} + \frac{15}{2} v \right) dv = \left(-\frac{15}{4v^2} + \frac{15}{4} v^2 \right) \Big|_1^2$

~~$\left(\frac{15}{2} \cdot \frac{1}{16} + \frac{15}{2} \cdot \frac{1}{4} \right) - \left(\frac{15}{2} \cdot \frac{1}{4} + \frac{15}{2} \cdot \frac{1}{16} \right) = \frac{15}{16} + \frac{15}{8} - \frac{15}{8} - \frac{15}{32} = \frac{15}{16} + \frac{15}{16} = \frac{30}{16} = \frac{15}{8}$~~

$= \frac{225}{16}$.

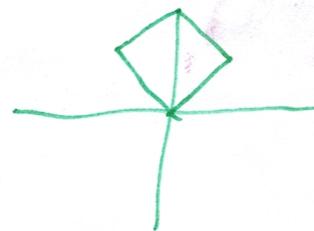
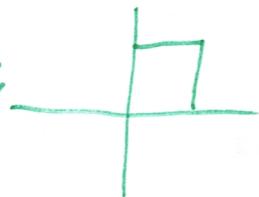
If $G: D \rightarrow R$, then $G^{-1}: D \rightarrow R$, the inverse of G , satisfies $G^{-1}(G(D)) = D$. It undoes G . (This only makes sense when G is one-to-one.)

Fact: $\text{Jac}(G^{-1}) = \frac{1}{\text{Jac}(G)}$. (so long as $\text{Jac}(G) \neq 0$.)

We can use this to simplify problems.

Ex: Let R be the square with vertices $(0, 0)$, $(-1, 1)$, $(1, 1)$, and $(0, 2)$. Calculate $\iint_R x^2 - y^2 dA$.

Sol: we can pick D in the uv -plane. We choose something easy, $[0, 1] \times [0, 1]$.



We want: $-G^{-1}(0,0) = (0,0)$. This will be true if our map is linear.

$$-G^{-1}(-1,1) = (0,1). \quad -A+B=0, \quad -C+D=1.$$

Thus $A=B$, $D=1+C$.

$$-G^{-1}(1,1) = (1,0), \quad A+B=1, \quad C+D=0. \text{ Thus}$$

$$C=-D, \text{ so } D=1-D, \quad 2D=1, \quad D=\frac{1}{2}, \quad C=-\frac{1}{2}.$$

$$A=1-B, \quad B=1-B, \quad 2B=1, \quad B=\frac{1}{2}, \quad A=\frac{1}{2}.$$

Therefore $G^{-1}(x,y) = \left(\frac{x+y}{2}, \frac{y-x}{2}\right)$. By the change of variables formula,

$$\iint_R f(x,y) dA = \iint_{S_0} f(x(u,v), y(u,v)) |Jac(G)| du dv.$$

$$|Jac(G)| = \frac{1}{|Jac(G^{-1})|} = \frac{1}{\frac{1}{2}} = 2.$$

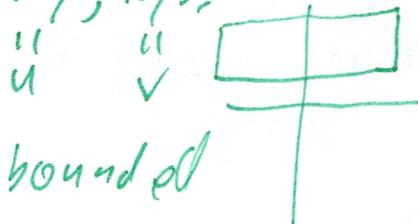
$$x^2 + y^2 = (x+y)(x-y) = 2u \cdot 2v = 4uv. \text{ Thus,}$$

$$\iint_{S_0} f(x,y) dA = \iint_{S_0} f(2u, 2v) |Jac(G)| du dv = \iint_{S_0} f(2u, 2v) 2 du dv = \iint_{S_0} f(2u, 2v) du dv = -2.$$

Ex: Compute $\iint_R xy(x^2+y^2) dA$ over R defined via

$$-3 \leq x^2 - y^2 \leq 3, \text{ and } 1 \leq xy \leq 4.$$

Sol: Let $G^{-1}(x,y) = (x^2 - y^2, xy)$.



Then $G^{-1}(R)$ is bounded

by $-3 \leq u \leq 3$ and $1 \leq v \leq 4$.

Thus
$$\iint_R f(x,y) dA = \iint_{G^{-1}(R)} f(x(u,v), y(u,v)) |Jac(G)| du dv$$

$$|\text{Jac}(G)| = \frac{1}{|\text{Jac}(G^{-1})|} = \frac{1}{|\det \begin{pmatrix} 2x & -2y \\ y & x \end{pmatrix}|} = \frac{1}{|2x^2 + 2y^2|}.$$

Thus

$$\begin{aligned} \iint_{G^{-1}(R)} f(x(u,v), y(u,v)) |\text{Jac}(G)| \, du \, dv &= \\ \int_1^4 \int_{-3}^3 v \cdot (x^2 + y^2) \cdot \frac{1}{2(x^2 + y^2)} \, du \, dv &= \int_1^4 \int_{-3}^3 \frac{v}{2} \, du \, dv = \int_1^4 3v \, dv \\ &= \left. \frac{3}{2} v^2 \right|_1^4 = 24 - \frac{3}{2} = \frac{45}{2}. \end{aligned}$$

Change of variables can be done just the same in \mathbb{R}^3 , where the Jacobian is

$$\det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}.$$

Ex: Let $S(\rho, \theta, \phi) = (\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi)$. Then

$$|\text{Jac}(G)| = \begin{vmatrix} \sin \phi \cos \theta & -\rho \sin \phi \sin \theta & \rho \cos \phi \cos \theta \\ \sin \phi \sin \theta & \rho \sin \phi \cos \theta & \rho \cos \phi \sin \theta \\ \cos \phi & 0 & -\rho \sin \phi \end{vmatrix} =$$

$$\begin{aligned} & \left| \cos \phi (-\rho^2 \sin \phi \cos \phi \sin^2 \theta - \rho^2 \sin \phi \cos \phi \cos^2 \theta) \right. \\ & - 0 \cdot (\rho \sin \phi \cos \phi \sin \theta \cos \theta - \rho \sin \phi \cos \phi \sin \theta \cos \theta) \\ & \left. + (-\rho \sin \phi)(\rho \sin^2 \phi \cos^2 \theta + \rho \sin^2 \phi \sin^2 \theta) \right| = \\ & \left| \cos \phi (-\rho^2 \sin \phi \cos \phi) + (-\rho \sin \phi)(\rho \sin^2 \phi) \right| = \end{aligned}$$

$$\left| -\rho^2 \sin \phi \cos^2 \phi - \rho^2 \sin^3 \phi \right| = \left| -\rho^2 \sin \phi \right| = \rho^2 \sin \phi.$$