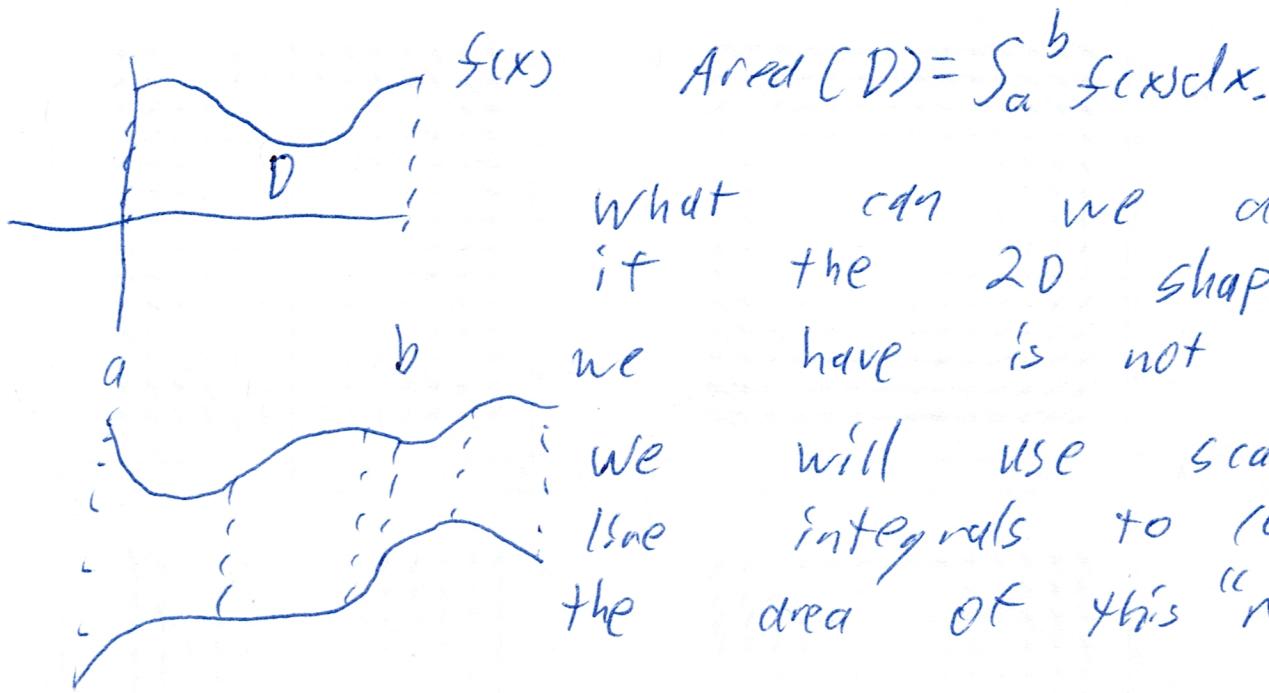


Scalar Line Integrals

Recall that in Calc 1, we calculated areas using integrals:



What can we do if the 2D shape we have is not flat? We will use scalar integrals to compute the area of this "ribbon."

Let $r(t) = (x(t), y(t), z(t))$ for $a \leq t \leq b$ be a parametric curve, (I.e., a function moving through 3D-space whose components are given by functions $x(t)$, $y(t)$, and $z(t)$). Then, if C is the path defined by r ,

$$\begin{aligned} \int_C f ds &= \int_a^b f(x(t), y(t), z(t)) \|r'(t)\| dt \\ &= \int_a^b f(x(t), y(t), z(t)) \cdot \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} dt \end{aligned}$$

Note that the direction and parameterization of r does not affect the area we are calculating, so changes do not affect the integral.

The $\|r(t)\|$ comes from the length of a tiny sliver of C . (I.e., if we straighten out C , $\|r'(t)\|dt = dx$.) It is like the one variable Jacobian,

Ex: $r(t) = (t^2, 3t^2)$, $0 \leq t \leq 4$. Calculate $\int_C x+ys$
 Sol: $r'(t) = (2t, 6t)$, $\|r'(t)\| = \sqrt{4t^2 + 36t^2} = \sqrt{40t^2} = 2\sqrt{10}t$.

Thus $\int_C x+ys = \int_0^4 4t^2 + 2\sqrt{10}t dt = \int_0^4 8\sqrt{10}t + 3t^4 dt = 2\sqrt{10}t^4 \Big|_0^4 = 512\sqrt{10}$,

Ex: Let C be the unit circle path traversed 1.5 times from $(1, 0)$ to $(-1, 0)$. Calculate $\int_C x^2 ds$.

Sol: ~~$r(t) = (\cos t, \sin t)$~~ , $0 \leq t \leq 3\pi$,
 $r'(t) = (-\sin t, \cos t)$
 $\|r'(t)\| = \sqrt{\sin^2 t + \cos^2 t} = \sqrt{1} = 1$

$$\int_C x^2 ds = \int_0^{3\pi} \cos^2 t dt = \left(\frac{t}{2} + \frac{\sin 2t}{4} \right)_0^{3\pi} = \frac{3\pi}{2}.$$

Recall that $\text{Area}(D) = \iint_D 1 dA$
 $\text{Volume}(E) = \iiint_E 1 dV$.

Similarly, the length of some curve C is given by $\int_C 1 ds$.

Ex: Find the length of the helix
 $r(t) = (2 \cos(2t), 2 \sin(2t), 3t)$, $0 \leq t \leq 4\pi$

$$r(t) = (2 \cos(2t), 2 \sin(2t), 3t), \quad 0 \leq t \leq 4\pi$$

$$S_0(1) \quad r'(+) = \langle -4\sin(2t), 4\cos(2t), 3 \rangle$$

$$\|r'(t)\| = \sqrt{16\sin^2(2t) + 16\cos^2(2t) + 4} = \sqrt{16+4} = \sqrt{20} = 2\sqrt{5}$$

$$\text{Length}(r(t)) = \int_0^{4\pi} 5 dt = 20\pi$$

Notice that for $y = f(x)$, we can represent its graph as the parametric curve $r(x) = (x, f(x))$. ($r(t) = (t, f(t))$). Then $r'(x) = \langle 1, f'(x) \rangle$, so $\|r'(x)\| = \sqrt{1 + f'(x)^2}$. Similarly, if $x = g(y)$, then we can consider the parametric curve $r(y) = (g(y), y)$. $r'(y) = \langle g'(y), 1 \rangle$, and $\|r'(y)\| = \sqrt{1 + g'(y)^2}$. We can use this to compute the arc length.

Ex: Let $y = 3(2+x)^{3/2}$, calculate the length of the graph for x between -2 and 3.

Sol: Let $r(x) = (x, 3(2+x)^{\frac{3}{2}})$. Then

$$r'(+) = \left\langle 1, d_{\frac{1}{2}} \sqrt{2+x} \right\rangle,$$

$$\|r'(t)\| = \sqrt{1 + \frac{81}{4}(2+t)} = \sqrt{\frac{83}{2} + \frac{81t}{4}}$$