

Vector Line Integrals

Recall that work is force acting over distance, I.e., $W = F \cdot d$. If the vector field F represents the force acting on a particle at a given point, then the work done by the vector field is obtained by adding up the work along rays, the path of the particle.

Recall that force is a vector pointing in the direction the force is acting. To find the component of a force $\langle a, b, c \rangle$ in the direction of the unit vector \hat{u} (i.e. $\|\hat{u}\|=1$), we take the dot product.

$$\begin{array}{c} \text{F} = \langle a, b, c \rangle \\ \text{F} \cdot \hat{u} \end{array}$$

So if we have a particle traveling along rays through the vector field F , the contribution of F to the particles movement is the component of $F(r(t))$ in the direction of $r'(t)$. The unit vector in the direction of $r'(t)$ is given by $\frac{r'(t)}{\|r'(t)\|}$. Thus the infinitesimal work done by F at $r(t)$ is

$$(F(r(t)) \cdot \frac{r'(t)}{\|r(t)\|}) \cdot ds = F(r(t)) \cdot \frac{r'(t)}{\|r(t)\|} \cdot \|r'(t)\| dt =$$

↑ ↑
Force distance

$F(r(t)) \cdot r'(t) dt$. Thus, the total work done by the vector field is given by

$$\int_C F \cdot \frac{r'(t)}{\|r(t)\|} ds = \int_a^b F(r(t)) \cdot r'(t) dt (= \int_C F \cdot dr)$$

We call this the vector line integral of F over C , (or r)

Ex: Let $F(x, y, z) = \langle 3, 2, z \rangle$, and let $r(t) = \langle t, t^2, t+2 \rangle$ for $0 \leq t \leq 5$. Compute $\int_C F \cdot dr$, where C is the path traversed by r .

Sol: $\int_C F \cdot dr = \int_0^5 \langle 3, 2, t+2 \rangle \cdot \langle 1, 2t, 1 \rangle dt =$
 $\int_0^5 3 + 4t + t+2 dt = \int_0^5 5t+5 dt = (\frac{5t^2}{2} + 5t) \Big|_0^5 = \frac{175}{2}$

It is important to stress that this is the work done by the vector field, not the total work done.

Ex: Find $\int_C F \cdot dr$, where $F(x, y) = \langle -y, x \rangle$ and $r(t) = \langle t, t \rangle$ for $a \leq t \leq b$, (C given by r)

Sol: $\int_C F \cdot dr = \int_a^b \langle -t, t \rangle \cdot \langle 1, 1 \rangle dt = \int_a^b -t + t dt = 0$

This occurs because $r'(t) = \langle 1, 1 \rangle$ is always perpendicular to $\langle -t, t \rangle$, so the vector field is not contributing to the work done.

Given a curve C going from (a, b, c) to (d, e, f) , there are infinitely many ways to ~~parametrize~~ parametrize it. Because of the chain rule, it turns out that, as with the scalar line integrals, the choice of parametrization does not matter. However, given C , then there is another path to consider: $-C$, the path starting at (d, e, f) and traveling backwards to reach (a, b, c) . Then

$$\int_C \mathbf{F} \cdot d\vec{r} = - \int_{-C} \mathbf{F} \cdot d\vec{r}$$

Ex: $\mathbf{F}(x, y, z) = \langle 3, 2, z^2 \rangle$, $r(t) = (9-t, 15-t, t)$, $0 \leq t \leq 5$.

Sol: Notice that $r(t) = \cancel{s}(5-t)$, where s is the path from the first example. Thus r and s define the same path, but traveling in opposite directions. Therefore, having r define C and s define $-C$

$$\int_C \mathbf{F} \cdot d\vec{r} = - \left(\int_{-C} \mathbf{F} \cdot d\vec{r} \right) = - \left(\frac{175}{2} \right) = -\frac{175}{2}$$

We verify using a direct computation:

$$\int_C \mathbf{F} \cdot d\vec{r} = \int_0^5 \langle 3, 2, 7-t \rangle \cdot \langle -1, 2t-10, 1 \rangle dt =$$

$$\begin{aligned} \int_0^5 -3+4t-20+t-7dt &= \int_0^5 5t-30 dt = \frac{5t^2}{2} - 30t \Big|_0^5 = \frac{125}{2} - 150 \\ &= \frac{125-300}{2} = -\frac{175}{2} \end{aligned}$$

If $F(x,y,z) = \langle P(x), Q(y), R(z) \rangle$, then we can write $\int_C F \cdot d\vec{r}$ as $\int_C P dx + Q dy + R dz$.

If C goes from (a,b) to (c,d,e) , then this would be $\int_a^c P(x) dx + \int_b^d Q(y) dy + \int_c^e R(z) dz$.

This is because of the chain rule: If

$r(t) = (x(t), y(t), z(t))$, then $\int_C F \cdot d\vec{r} =$

$$\int_a^b \langle P(x(t)), Q(y(t)), R(z(t)) \rangle \cdot \langle x'(t), y'(t), z'(t) \rangle dt$$

$$= \int_a^b P(x(t)) x'(t) dt + \int_a^b Q(y(t)) y'(t) dt + \int_a^b R(z(t)) z'(t) dt$$

$$= \int_a^b P(x) dx + \int_b^c Q(y) dy + \int_c^d R(z) dz,$$

Ex: $F(x,y,z) = \langle x^2, y, -z^3 \rangle$, $r(t) = \langle t, t^2, t^3 \rangle$ (compute $0 \leq t \leq 2$)

$$\int_C x^2 dx + y dy + (-z^3) dz,$$

$$\text{Sol: } \int_C x^2 dx + y dy + (-z^3) dz = \int_0^2 x^2 dx + \int_0^4 y dy + \int_0^8 -z^3 dz =$$

$$\frac{x^3}{3} \Big|_0^2 + \frac{y^2}{2} \Big|_0^4 - \frac{z^4}{4} \Big|_0^8 = \frac{8}{3} + 8 - 1024.$$

Alternatively,

$$\begin{aligned} \int_C F \cdot d\vec{r} &= \int_0^2 \langle t^2, t^2, -t^3 \rangle \cdot \langle 1, 2t, 3t^2 \rangle dt = \\ &= \int_0^2 t^2 + 2t^3 - 3t^5 dt = \left(\frac{t^3}{3} + \frac{t^4}{2} - \frac{t^6}{4} \right) \Big|_0^2 \\ &= \frac{8}{3} + 8 - 1024 \end{aligned}$$