

## Conservative Vector Fields

We saw before the exam that a conservative vector field  $\mathbf{F}$  is such that, for any simple, closed  $C$ ,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

We also saw that  $\mathbf{F}$  is conservative exactly when there is a potential function  $f$ , i.e. a scalar function with

$$\mathbf{F} = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \nabla f$$

Notice that potential functions are unique up to a constant. If  $f$  and  $g$  are potential functions, i.e.  $\nabla f = \nabla g \approx \mathbf{F}$ , then  $\nabla f - \nabla g = \mathbf{F} - \mathbf{F} = \langle 0, 0, 0 \rangle$ . So  $\nabla f - \nabla g = \langle f_x - g_x, f_y - g_y, f_z - g_z \rangle = \langle 0, 0, 0 \rangle$ . If  $f-g$  depended on some variable  $x, y$ , or  $z$ , then the corresponding partial derivative would not be zero.

A region in space (2D or 3D) is connected if there is a path between any two points.

Ex:

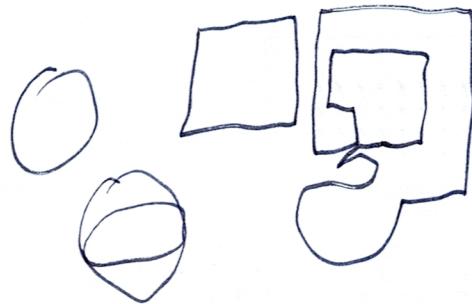


Non-ex:



A region  $D$  is simply connected if any simple, closed curve in  $D$  can be shrunk to a point without leaving  $D$ . (And  $D$  is connected.) I.e.,  $D$  has no holes.

Ex:



Non-ex:



A vector field  $F$  is conservative when its domain is open and simply connected and its cross partials agree, i.e.  $P_y - Q_x = P_2 - R_x = Q_z - R_y = 0$ . (We saw the reverse last week, i.e. ~~if F's cross partials do not agree, then it is not conservative.~~)

This enables us to check if a vector field is conservative by ensuring there are no holes in the domain and checking the cross partials. If both succeed, then we know there is a potential function. But how do we find it?

We illustrate the algorithm for finding  $f$  with an example.

Ex: Find a potential function for

$$F(x,y,z) = \langle 2x\cos yz, -x^2yz\sin(yz) + yz, -\sin(yz)x^2y + \frac{y^2}{2} + 1 \rangle$$

Sol: If  $F = \nabla f$ , then we have

$$\begin{aligned} & \langle 2x\cos yz, -x^2yz\sin(yz) + yz, -\sin(yz)x^2y + \frac{y^2}{2} + 1 \rangle = \\ & \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle. \end{aligned}$$

Then  $\frac{\partial f}{\partial x} = 2x\cos yz$  so

$$f = \int 2x\cos yz dx = x^2\cos yz + C$$

However,  $C$  is constant with respect to  $x$ , so this is  $x^2\cos yz + g(y,z)$ . This gives us a new equation:

$$\begin{aligned} \frac{\partial f}{\partial y} &= -x^2yz\sin(yz) + yz \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2\cos yz + g(y,z)) \\ &= -x^2yz\sin(yz) + \frac{\partial}{\partial y}g(y,z) \end{aligned}$$

Thus  $\frac{\partial}{\partial y}g(y,z) = yz$ , so

$$g(y,z) = \int yz dy = \frac{y^2}{2}z + h(z).$$

Thus  $f(x,y,z) = x^2\cos yz + \frac{y^2}{2}z + h(z)$ . Finally, we do the same for  $z$ :

$$\frac{\partial f}{\partial z} = -x^2yz\sin(yz) + \frac{y^2}{2} + 1 \quad \text{and} \quad \frac{\partial f}{\partial z} = -x^2yz\sin(yz) + \frac{y^2}{2} + h'(z)$$

Thus  $h'(z) = 1$ , so  $h(z) = \int 1 dz = z + C$ .

thus  $f(x,y,z) = x^2\cos yz + \frac{y^2}{2}z + z + C$  for any  $C$ .

Thus, once we determine  $F$  is conservative, we have the following algorithm to find its potential function:

1. Integrate the first component with respect to  $x$  to get

$$f(x, y, z) = SPdx + g(y, z)$$

2. Set  $\frac{\partial f}{\partial y} = Q$  and  $\frac{\partial f}{\partial x} = \frac{\partial}{\partial y}(SPdx) + \frac{\partial}{\partial y}g(y, z)$  equal to one another to get  $\frac{\partial}{\partial y}g(y, z) = m(y, z)$

3. Integrate to get

$$g(y, z) = \int m(y, z) dy, \text{ and thus}$$

$$f(x, y, z) = SPdx + \int m(y, z) dy + h(z)$$

4. Set  $\frac{\partial f}{\partial z} = R$  and  $\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}(SPdx + \int m(y, z) dy) + h'(z)$

$$+ h'(z),$$

simplify to find  $h'(z) = h(z)$ .

5. Integrate to get

$$f(x, y, z) = SPdx + \int m(y, z) dy + \int h(z) dz + C.$$

Ex: Find a potential function for

$$F(x, y, z) = \langle 2x \ln y, \frac{x^2}{y} + z^2, 2yz \rangle.$$

Sol:

$f = \int 2x \ln y dx = x^2 \ln y + g(y, z)$	$\frac{\partial f}{\partial z} = 2yz \text{ and}$
$\frac{\partial f}{\partial y} = \frac{x^2}{y} + z^2 \text{ and } \frac{x^2}{y} + \frac{\partial}{\partial y}g(y, z)$	$\frac{\partial f}{\partial z} = 2yz + h'(z),$
$g(y, z) = \int z^2 dy = z^2 y + h(z)$	$\text{so } h'(z) = 0$
	and $h(z) = C.$