

## Green's Theorem

positively oriented

Green's Theorem! Let  $C$  be a simple, closed curve with interior  $D$ , and let  $\mathbf{F}(x,y)$  be a vector field defined on  $D$ . Then

$$\oint_C \mathbf{F} \cdot d\vec{r} = \oint_C P dx + Q dy = \iint_D Q_x - P_y dA$$

Note: If  $C$  is negatively oriented, add a minus sign to get  $\oint_C \mathbf{F} \cdot d\vec{r} = \iint_D P_y - Q_x dA$ .

Ex:  $\mathbf{F}(x,y) = \langle x^2y, x-3 \rangle$ ,  $C$  the rectangular path traversing  $(1,1), (4,1), (4,5)$ , and  $(1,5)$ . Compute  $\oint_C \mathbf{F} \cdot d\vec{r}$ .

Sol: By Green's Theorem,

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\vec{r} &= \iint_D Q_x - P_y dA = \int_1^4 \int_1^5 0 - x^2 dy dx \\ &= \int_1^4 \int_1^5 -x^2 dy dx \\ &= \int_1^4 -4x^2 dx \\ &= -4 \left[ x^3 \right]_1^4 = -84 \end{aligned}$$

We will see next week that  $Q_x - P_y$  is the "curl" of  $\mathbf{F}$ , a measurement of how nonconservative  $\mathbf{F}$  is. So Green's Theorem is essentially adding up how nonconservative  $\mathbf{F}$  is over  $C$ .

You can double check the previous example directly by parametrizing the rectangular path and computing a vector line integral.

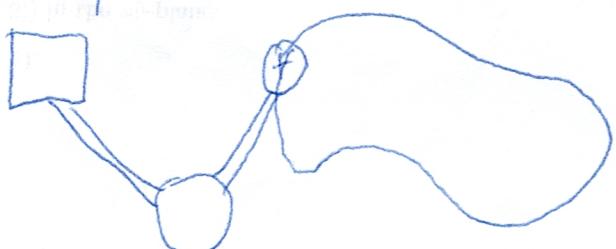
We can use Green's Theorem to compute area: If  $F(x,y) = \langle P, Q \rangle$  is such that  $Q_x - P_y = 1$ , then Green's Theorem says that  $\iint_D 1 dA = \oint_C F \cdot d\vec{r}$ . There are lots of such fields:  $\langle 0, x \rangle$ ,  $\langle y, 2x \rangle$ , etc.

Ex: Calculate the area of the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$ .

Sol: This can be parametrized by  $r(t) = (a \cos t, b \sin t)$ ,  $0 \leq t \leq 2\pi$ . Then

$$\begin{aligned}\iint_D 1 dA &= \iint_D \frac{\partial}{\partial x} x - \frac{\partial}{\partial y} 0 dA = \oint_C \langle 0, x \rangle \cdot d\vec{r} \\ &= \int_0^{2\pi} \langle 0, a \cos t \rangle \cdot \langle -a \sin t, b \cos t \rangle dt \\ &= \int_0^{2\pi} ab \cos^2 t dt = ab \left( \frac{t}{2} + \frac{\sin 2t}{4} \right) \Big|_0^{2\pi} \\ &= ab\pi\end{aligned}$$

A planimeter uses the principles of Green's Theorem to compute the area of arbitrary shapes. It computes the area by adding up  $F \cdot d\vec{r}$ .

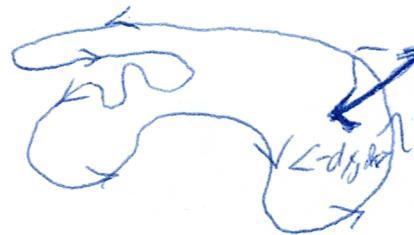


## Flux Form of Green's Theorem

$$\oint_C (F \cdot \vec{N}) ds = \oint_C (-Q dx + P dy)$$

$$= \iint_D P_x + Q_y dA$$

if  $C$  is counterclockwise. (Add a minus sign if it is clockwise.)



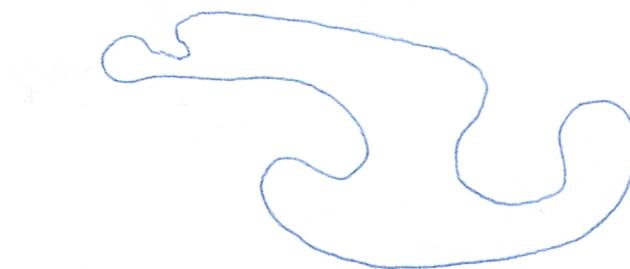
We will see that  $P_x + Q_y$  is the "divergence" of  $F$ , which measures out flow at a given point. The flux, the amount of stuff crossing the boundary, is obtained by adding up all the stuff flowing out of points inside.

Ex: Let  $F = \langle x, y \rangle$ . Calculate the flux through the unit circle.

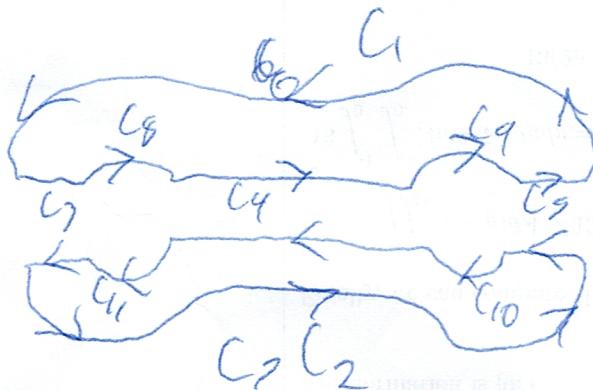
$$\text{Sol: } \oint_C (F \cdot \vec{N}) ds = \oint_C (-y dx + x dy)$$

$$= \iint_D 1 + 1 dA = 2 \iint_D dA = 2\pi$$

So far, Green's Theorem applies to regions enclosed by simple closed curves.



However, we can apply it to regions that are not simply connected:



$$\iint_D Q_x - P_y \, dA = \oint_{C_1} F \cdot d\vec{r} +$$

$$\oint_{C_2} F \cdot d\vec{r} = \oint_{C_4} F \cdot d\vec{r} - \oint_{C_4} F \cdot d\vec{r}$$

$$= \oint_{C_6} F \cdot d\vec{r} + \oint_{C_7} F \cdot d\vec{r} + S_{C_9} + S_{C_4} + S_{C_8}$$

$$+ S_{C_{11}}$$

