

# Curl and Divergence:

$\nabla$  represents the "vector"  $\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$ .  
We've seen this before with  $\nabla f =$   
 $\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$ .

The curl of a vector field  $F$ , denoted by  $\text{curl}(F)$ , is defined via " $\nabla \times F$ ", where we take derivatives instead of multiplying. That is,

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \hat{i} - \left( \frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) \hat{j} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{k}$$

Notice that  $\text{curl}(F) = \vec{0}$  exactly when the mixed partials are equal. So  $\text{curl}(F) = \vec{0}$  whenever  $F$  is conservative, and if the domain of  $F$  is simply connected, then  $\text{curl}(F) = \vec{0}$  implies  $F$  is conservative.

Notice that if  $F(x,y) = \langle P, Q \rangle$ , then we can treat this as  $F(x,y,z) = \langle P, Q, R \rangle$  where  $R=0$ . Then

$$\text{curl}(F) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = (0) \hat{i} - (0) \hat{j} + (Q_x - P_y) \hat{k}$$

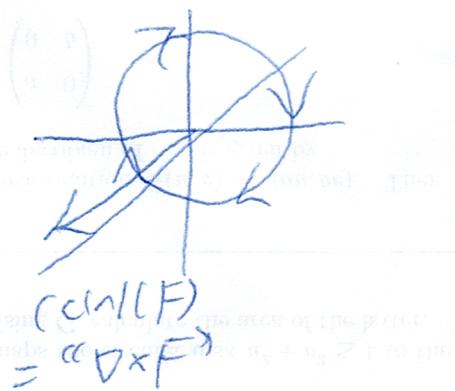
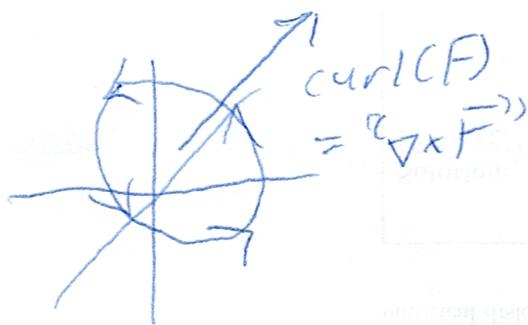
This was what we saw curl ds in Green's Theorem.

Ex:  $F(x,y,z) = \langle xz, xyz, -y^2 \rangle$ . (compute  $\text{curl}(F)$ )

Sol: 
$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix} = (-2y - xy)\hat{i} - (0 - x)\hat{j} + (yz - 0)\hat{k}$$
$$= \langle -2y - xy, x, yz \rangle$$

Curl measures how much a vector field rotates around a point.

The direction of  $\text{curl}(F)$  is perpendicular to the plane of rotation and whether it is "up" or "down" (i.e., it follows the right hand rule) depends on if it is rotating clockwise or counter clockwise.



We shall see that there is a 3D-version of Green's Theorem - Stokes' Theorem.

Similarly, the divergence of  $F$  is " $\nabla \cdot F$ ",

i.e.  $\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$ .

In 2D, this is  $\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle \cdot \langle P, Q \rangle = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$ .

This is the quantity we saw in

the flux form of Green's Theorem.

Ex:  $F(x, y, z) = \langle xz, xyz, -y^2 \rangle$ . (compute  $\text{div}(F)$ )

$$\text{div}(F) = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle xz, xyz, -y^2 \rangle =$$

$$\begin{aligned} \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(-y^2) &= z + xz + 0 \\ &= z + xz \end{aligned}$$

A point  $(x, y, z)$  is called a source

if  $\text{div}(F) > 0$  at  $(x, y, z)$ . It is

called a sink if  $\text{div}(F) < 0$ . If

$\text{div}(F) = 0$  for all points in the

domain, then  $F$  is called incompressible.

A source represents water flowing in (more water flowing from the point

than out), and a sink represents water flowing out. (more water coming into the water than out.)

Notice that, if  $F$  is any nice vector field,

$$\operatorname{div}(\operatorname{curl}(F)) = \operatorname{div} \left( \begin{matrix} \frac{\partial R}{\partial x} - \frac{\partial Q}{\partial y} \\ \frac{\partial P}{\partial y} - \frac{\partial R}{\partial z} \\ \frac{\partial Q}{\partial z} - \frac{\partial P}{\partial x} \end{matrix} \right) =$$

$$\operatorname{div} \left( \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \hat{i} - \left( \frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) \hat{j} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{k} \right)$$

$$= \frac{\partial R}{\partial y \partial x} - \frac{\partial Q}{\partial z \partial x} + \frac{\partial P}{\partial z \partial y} - \frac{\partial R}{\partial x \partial y} + \frac{\partial Q}{\partial x \partial z} - \frac{\partial P}{\partial y \partial z}$$

$$= 0 \quad \text{if the mixed partials}$$

are the same. (i.e.,  $P, Q, R$  second differentiable and continuous.)

As long as the domain is simply connected, the converse is true. If domain of  $G$  is simply connected and  $\operatorname{div}(G) = 0$ , then there is some

vector field  $F$  with  $\operatorname{curl}(F) = G$ .

This is similar to gradient fields and conservative vector fields and curl over simply connected domains.