

Parametrized Surfaces

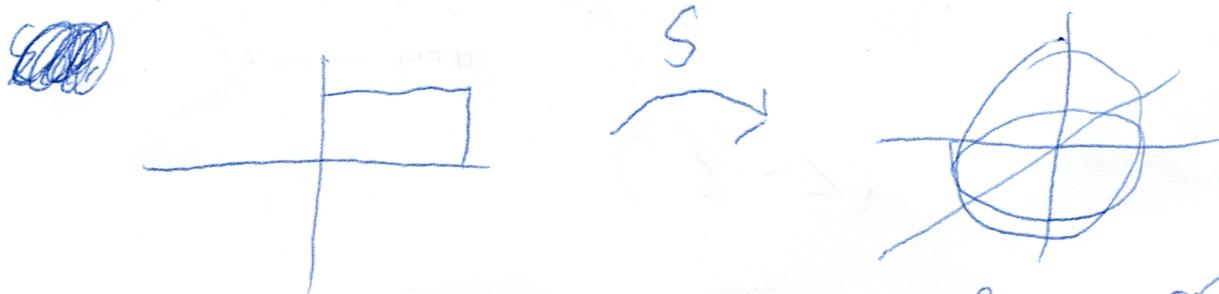
A parametrized surface is of the form

$$R(s, t) = (x(s, t), y(s, t), z(s, t))$$

for (s, t) in some region D in the 2D-plane, called the parameter domain.

In other words, we are describing points in 3d using two variables like we did with one variable for parametric curves.

Ex: $S(\theta, \phi) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)$,
 θ, ϕ in $[0, 2\pi] \times [0, \pi]$



S represents the surface of the unit sphere. We can get the sphere of radius R by multiplying each component by R .

Ex: Given $z = f(x, y)$, we can parametrize the graph over D via $R(x, y) = (x, y, f(x, y))$, for (x, y) in D .

For example, $(x, y, x^2 + y^2)$

Ex: Give a parametrization of the cone $z = x^2 + y^2$ inside the cylinder $x^2 + y^2 \leq 4$.

Sol: The upper half of the cone is $z = \sqrt{x^2 + y^2}$, and the lower half is $z = -\sqrt{x^2 + y^2}$. In polar coordinates, this is $z = r$ and $z = -r$ for $0 \leq r \leq 2$.

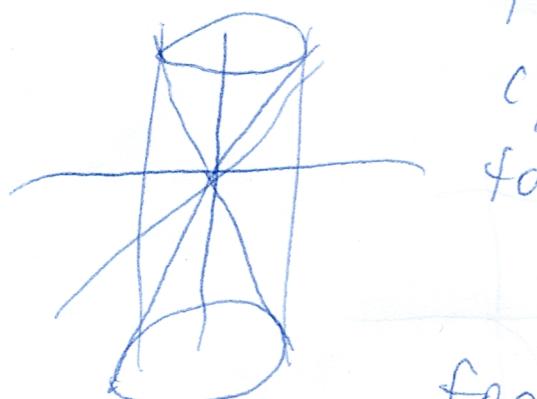
Thus we can modify cylindrical coordinates

to get $R(r, \theta) =$

$$(r \cos \theta, r \sin \theta, r)$$

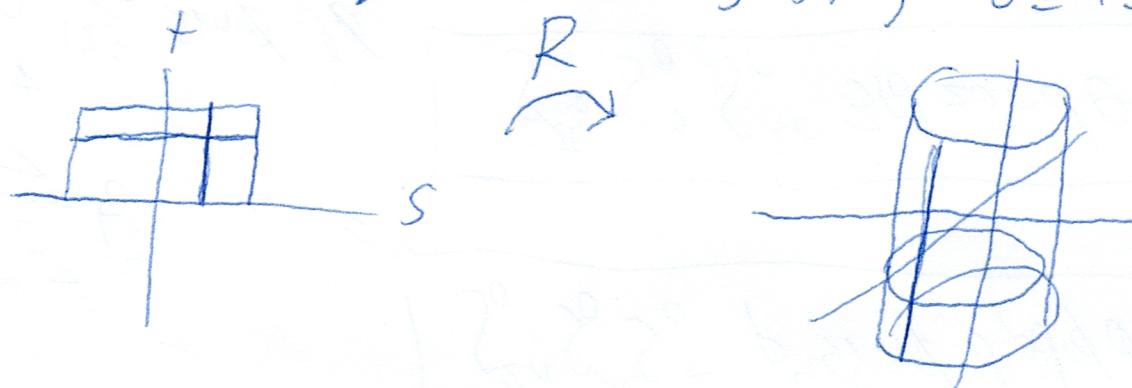
for $-2 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$.

In cylindrical coordinates, r cannot be negative. However, when parametrizing we can do what we want.



Like we did with change of variables,
we look at the image of lines under
parametrization to see the behavior
of our shape.

Ex: $R(s, t) = (3 \cos t, 3 \sin t, s)$, $0 \leq t \leq 2\pi$, $-2 \leq s \leq 2$



If $s=c$, then $R(c, t)$ is the circle of radius 3 at height c .

If $t=c$, then $R(s, c)$ is the vertical line at angle c .

These are called the grid curves of the shape/surface.



As we do with all integrals,
we want to break the surface
down into small intervals for the

purposes of calculating a Riemann sum,
Note that the grid curves cut
the region into smaller ones,



As the distance between grid curves goes to zero, this becomes a parallelogram.

The sides of this parallelogram are the tangent vectors to the grid curves.
I.e., if $R(x,y) = (x(x,y), y(x,y), z(x,y))$, then

$$T_x = \left\langle \frac{\partial x}{\partial s}, \frac{\partial y}{\partial s}, \frac{\partial z}{\partial s} \right\rangle \text{ and}$$

$$T_y = \left\langle \frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t} \right\rangle$$

The area of this parallelogram can be computed using the cross product and the normal vector!

$$N = T_x \times T_y, \|N\| \text{ is the area of the parallelogram.}$$

The parameterization is called regular if $\|N\|$ is never zero.

$$\text{Ex: } R(x,y) = \langle x, y, f(x,y) \rangle$$

$$T_x = \left\langle 1, 0, \frac{\partial f}{\partial x} \right\rangle, T_y = \left\langle 0, 1, \frac{\partial f}{\partial y} \right\rangle$$

$$N = T_x \times T_y = \left\langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \right\rangle. \text{ Thus}$$

graph surfaces are always regular.