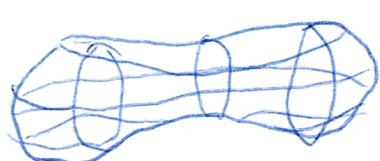


Scalar Surface Integrals

Recall that we took a parameterized surface $R(s, t)$ and cut it into pieces using grid curves.



The area of these small pieces was calculated via

$$T_s = \left\langle \frac{\partial x}{\partial s}, \frac{\partial y}{\partial s}, \frac{\partial z}{\partial s} \right\rangle$$

$$T_t = \left\langle \frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t} \right\rangle$$

$$N = T_s \times T_t$$

$\|N\| = \text{area of the piece.}$

Then, if S is the surface defined by R , we represent the scalar surface integral of f over S via

$$\iint_S f(x, y, z) dS = \iint_D f(R(s, t)) \|N(s, t)\| dA$$

↑ "dA, dV, ds"

The length of the normal vector is our scaling factor like the Jacobian or length of the tangent vector.

Ex: Calculate the surface area of the part of the cone $z^2 = x^2 + y^2$ above $x^2 + y^2 = 9$

So if we can parametrize S via

$$R(\theta, z) = (z \cos \theta, z \sin \theta, z) \quad 0 \leq z \leq 3, \quad 0 \leq \theta \leq 2\pi$$

$$T_\theta = \langle -z \sin \theta, z \cos \theta, 0 \rangle$$

$$T_z = \langle \cos \theta, \sin \theta, 1 \rangle$$

$$N = T_\theta \times T_z = \langle z \cos \theta, z \sin \theta, -z \rangle$$

$$\|N\| = \sqrt{z^2 \cos^2 \theta + z^2 \sin^2 \theta + z^2} = \sqrt{2z^2} = \sqrt{2} z$$

Then the surface area is given by

$$\begin{aligned} \iint_S 1 \, dS &= \iint_D 1 \cdot z \sqrt{2} \, dA = S_0^{2\pi} S_0^3 \sqrt{2} \int_0^3 z \, dz \, d\theta \\ &= S_0^{2\pi} \frac{\sqrt{2} z^2}{2} \Big|_0^3 = S_0^{2\pi} \frac{9}{2} \sqrt{2} \, d\theta = 9\sqrt{2}\pi \end{aligned}$$

Ex: Suppose a unit sphere has charge $3z^2$ at every point on its surface. Calculate the total charge.

Sol: $R(\theta, \phi) = (\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi), \quad 0 \leq \theta \leq 2\pi$

$$T_\theta = \langle -\sin \theta \sin \phi, \cos \theta \sin \phi, 0 \rangle$$

$$T_\phi = \langle \cos \theta \cos \phi, \sin \theta \cos \phi, -\sin \phi \rangle$$

$$N = \langle -\cos \theta \sin^2 \phi, -\sin \theta \sin^2 \phi, -\sin \phi \cos \theta \rangle$$

$$\|N\| = \sin \phi$$

$$\text{Charge} = S_0^{2\pi} \int_0^\pi 3 \cos^2 \theta \sin^2 \phi \, d\theta \, d\phi = \int_0^{2\pi} \frac{\cos^3 \theta}{3} \Big|_0^\pi \, d\theta = S_0^{2\pi} \frac{2}{3} \, d\theta = \frac{4\pi}{3}$$

If our parameterization is of the form $R(s,t) = \langle S_s + t f(S_s), g(S_s) \rangle$, then

$$T_s = \left\langle 1, 0, \frac{\partial S}{\partial s} \right\rangle$$

$$T_t = \left\langle 0, 1, \frac{\partial S}{\partial t} \right\rangle$$

$$N = \left\langle -\frac{\partial S}{\partial s}, -\frac{\partial S}{\partial t}, 1 \right\rangle$$

$$\|N\| = \sqrt{\frac{\partial S^2}{\partial s} + \frac{\partial S^2}{\partial t} + 1}$$

Ex: Compute $\iint_S 4 ds$, where S is the part of the paraboloid $z = 1 - x^2 - y^2$ with $z \geq 0$.

$$\text{Sol: } \|N\| = \sqrt{1+4x^2+4y^2}, \text{ so}$$

$\iint_S 4 ds = \iint_D 4 \sqrt{1+4x^2+4y^2} dA$, where D is the unit disk $x^2+y^2 \leq 1$, using polar coordinates,

$$\begin{aligned} \iint_D 4 \sqrt{1+4x^2+4y^2} dA &= \int_0^{2\pi} \int_0^1 4r \sqrt{1+4r^2} dr d\theta = \\ &\int_0^{2\pi} \frac{4}{3} \frac{(1+4r^2)^{3/2}}{3} \Big|_0^1 d\theta = \int_0^{2\pi} \frac{4}{3} \frac{5^{3/2}}{3} - \frac{1}{3} d\theta = \\ &\frac{5^{3/2}-1}{3}(2\pi) \end{aligned}$$

Just like with line integrals, the parametrization does not matter. If the parameter domain is smaller, then the space between grid curves is bigger. (I.e. the parameter domain is stretched further to make the surface S .) Similarly, if $\mathbf{N} = \mathbf{T}_t \times \mathbf{T}_s$ instead of $\mathbf{N} = \mathbf{T}_s \times \mathbf{T}_t$, then $\mathbf{N} = -\mathbf{N}$, so $\|\mathbf{N}\| = \|\mathbf{N}\|$.

Ex: Let S be $x^2 + y^2 \leq 9$, $z=0$.

Then we could parametrize this as $\mathbf{R}(r, \theta) = (r \cos \theta, r \sin \theta, 0)$, which gives

$$0 \leq r \leq 3 \quad \mathbf{T}_r = (\cos \theta, \sin \theta, 0)$$

$$0 \leq \theta \leq 2\pi \quad \mathbf{T}_\theta = (-r \sin \theta, r \cos \theta, 0)$$

$$\mathbf{N} = \langle 0, 0, 1 \rangle$$

$$\|\mathbf{N}\| = 1$$

Alternatively, $\mathbf{R}(r, \theta) = (3 \cos \theta, 3 \sin \theta, 0)$

$$0 \leq r \leq 1 \quad \mathbf{T}_r = \langle 3 \cos \theta, 3 \sin \theta, 0 \rangle$$

$$0 \leq \theta \leq 2\pi \quad \mathbf{T}_\theta = \langle -3 \sin \theta, 3 \cos \theta, 0 \rangle$$

$$\mathbf{N} = \langle 0, 0, 1 \rangle$$

$$\|\mathbf{N}\| = 3$$