

Vector Surface Integrals

Recall that, given a parametrization $R(u, v)$, we have two choices of normal vector: $N = T_u \times T_v$ and $N = T_v \times T_u$.

A closed surface (i.e. one with no boundary) is positively oriented if the normal vector points outwards.

A non-closed surface is positively oriented if, the surface normals are parallel to the body of someone walking along the boundary with the left side facing the surface. (Upwards for most surfaces)

In other words, the surface is positively oriented if its normals follow the right hand rule.



Given a vector field F and a surface S parametrized by $R(u, v)$, then the surface integral of F over S is

$$\iint_S F \cdot d\vec{S} = \iint_S (F \cdot \hat{N}) dS = \iint_D F \cdot N du dv$$

This only depends on the orientation (which changes the sign) but not the parametrization.

This is called the flux of F through (or across) S , and it represents flow of stuff through the surface. If flux is positive for a positively oriented surface, it represents stuff leaving. Negative flux represents stuff entering.

Ex: Calculate $\iint_S F \cdot d\vec{S}$, where $F = \langle -y, x, 0 \rangle$ and

S is given by $R(s, t) = \langle s, t^2 - s, s + t \rangle$, $0 \leq s \leq 3$
with upwards normals, $0 \leq t \leq 4$

Sol: $T_s = \langle 1, -1, 1 \rangle$

$T_t = \langle 0, 2t, 1 \rangle$

$N = T_s \times T_t = \langle -2t+1, -1, 2t \rangle$

This is the normal vector with upwards facing normals. Then

$$\iint_S F \cdot d\vec{S} = \int_0^3 \int_0^4 \langle s-t^2, s, 0 \rangle \cdot \langle -2t-1, -1, 2t \rangle dt ds$$

$$= \int_0^3 \int_0^4 -2st + 2t^3 - s + t^2 - s dt ds$$

$$= \int_0^3 \int_0^4 2t^3 + t^2 - 2st - s dt ds$$

$$= \int_0^3 \left[\frac{t^4}{2} + \frac{t^3}{3} - st^2 - 2st \right]_0^4 ds$$

$$= \int_0^3 \left(128 + \frac{64}{3} - 16s - 8s \right) ds = \left(128 + \frac{64}{3} \right) s - 12s^2 \Big|_0^3$$

$$= 448 - 108 = 340,$$

Ex: Calculate the flux of $F = \langle 0, -z, y \rangle$ through the portion of the unit sphere in the first octant, oriented outwards

Sol: Recall that the normal vector for a sphere of radius R pointing outwards is $R^2 \sin \phi \langle x, y, z \rangle$. Thus flux is

$$\begin{aligned} \iint_S F \cdot d\vec{S} &= \iint_S R^2 \sin \phi \langle x, y, z \rangle \cdot \langle 0, -z, y \rangle d\phi d\theta \\ &= \iint_S R^2 \sin \phi (-yz + zy) d\phi d\theta \\ &= \iint_S 0 d\phi d\theta = 0. \end{aligned}$$

Given units, flux computes physical quantities.

Ex: Suppose water is flowing with current $\langle 2x, 2y, 2z \rangle$ ~~centimeters~~ meters per second. How much water flows through the upper half of the sphere of radius 3?

Sol: $\iint_S \langle 2x, 2y, 2z \rangle \cdot 3 \sin \phi \langle x, y, z \rangle d\phi d\theta$

$$\begin{aligned} &= \iint_S 3 \sin \phi (2x^2 + 2y^2 + 2z^2) d\phi d\theta = \iint_S 54 \sin \phi d\phi d\theta \\ &= \int_0^{2\pi} \int_0^{\pi} 54 \sin \phi d\phi d\theta = \int_0^{2\pi} -54 \cos \phi \Big|_0^{\pi} d\theta = \int_0^{2\pi} 108 d\theta = 108 \cdot 2\pi = 216\pi \end{aligned}$$

Ex: Let S be the cap $x^2 + y^2 = 1$, $0 \leq z \leq 1$, with both top and bottom. Suppose the normals are inwards-facing. Find the flux of $F = \langle -y, z, x \rangle$ through S .

Sol: The top of the cap is parametrized by $R(r, \theta) = (r \cos \theta, r \sin \theta, 1)$, and has normal vector $\langle 0, 0, -r \rangle$. Then the flux through the top is

$$\int_0^{2\pi} \int_0^1 \langle -r \sin \theta, 1, r \cos \theta \rangle \cdot \langle 0, 0, -r \rangle dr d\theta = \int_0^{2\pi} \left. -\frac{r^3 \cos \theta}{3} \right|_0^1 d\theta = \int_0^{2\pi} -\frac{1}{3} \cos \theta d\theta = -\frac{1}{3} \sin \theta \Big|_0^{2\pi} = 0.$$

The bottom of the cap is similar, with normal $\langle 0, 0, r \rangle$ and flux 0.

Finally, the side is parametrized via

$$R(\theta, z) = (\cos \theta, \sin \theta, z), \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq z \leq 1.$$

$$\text{Then } T_\theta = \langle -\sin \theta, \cos \theta, 0 \rangle$$

$$T_z = \langle 0, 0, 1 \rangle$$

$$N = \langle \cos \theta, \sin \theta, 0 \rangle$$

This points outward, so we need $\langle -\cos \theta, -\sin \theta, 0 \rangle$ instead. Then the flux is

$$\begin{aligned} & \int_0^{2\pi} \int_0^1 \langle -\sin \theta, z, \cos \theta \rangle \cdot \langle -\cos \theta, -\sin \theta, 0 \rangle dz d\theta \\ &= \int_0^{2\pi} \int_0^1 (-\sin \theta \cos \theta + z \sin \theta) dz d\theta = \int_0^{2\pi} \left. -\sin \theta \cos \theta + \frac{z^2 \sin \theta}{2} \right|_0^1 d\theta \\ &= \left(\cos^2 \theta + \left(-\frac{\cos \theta}{2} \right) \right) \Big|_0^{2\pi} = 0. \end{aligned}$$