

Stokes' Theorem

Given a surface S , the edge is called the boundary, notated by ∂S .

Ex:



A surface is closed if the boundary is empty. The classic example is the sphere.

If S is a piecewise smooth, positively oriented surface whose boundary ∂S is a simple, closed curve, and F is a continuously differentiable vector field, then

$$\oint_{\partial S} F \cdot d\vec{r} = \iint_S \text{curl}(F) \cdot d\vec{S}$$

Recall that, for 2D vector fields,

$\text{curl}(F) = (Q_x - P_y) \hat{k}$. For a flat shape, $\hat{N} = \langle 0, 0, 1 \rangle$, so we can

see Stokes' Theorem is a more general form of Green's Theorem.

Ex: Let S be the graph of $z = 4 - x^2 - y^2$ with $z \geq 0$. Let $F = \langle y, 2z, x^2 \rangle$. Compute $\iint_S \text{curl}(F) \cdot d\vec{S}$

Sol: Notice that dS is the circle of radius 2 in the xy -plane. So, by Stokes' Theorem,

$$\begin{aligned} \iint_S \text{curl}(F) \cdot d\vec{S} &= \oint_{\partial S} F \cdot d\vec{r} = \int_0^{2\pi} \langle 2\sin\theta, 0, 4\cos^2\theta \rangle \cdot \langle -2\sin\theta, 2\cos\theta, 0 \rangle d\theta \\ &= \int_0^{2\pi} -4\sin^2\theta d\theta = -4 \left(\frac{\theta}{2} - \frac{\sin 2\theta}{4} \right) \Big|_0^{2\pi} = -4\pi \end{aligned}$$

We can double check this by computing directly:

$$\text{curl}(F) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & 2z & x^2 \end{vmatrix} = \langle -2, -2x, -1 \rangle$$

$$R(s, t) = \langle s \cos t, s \sin t, 4 - s^2 \rangle, \quad 0 \leq s \leq 2, \quad 0 \leq t \leq 2\pi$$

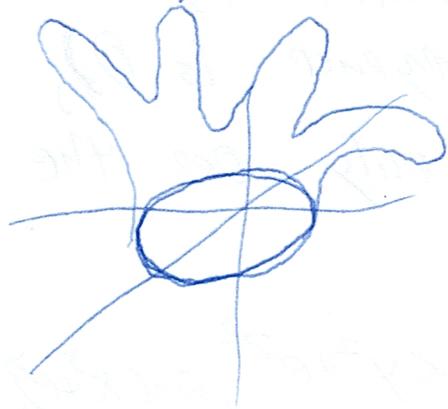
$$T_s = \langle \cos t, \sin t, -2s \rangle$$

$$T_t = \langle -s \sin t, s \cos t, 0 \rangle$$

$$N = \langle 2s^2 \cos t, 2s^2 \sin t, s \rangle$$

$$\begin{aligned} \iint_S \text{curl}(F) \cdot d\vec{S} &= \int_0^{2\pi} \int_0^2 \langle -2, -2s \cos t, -1 \rangle \cdot \langle 2s^2 \cos t, 2s^2 \sin t, s \rangle ds dt \\ &= \int_0^{2\pi} \int_0^2 -4s^2 \cos t - 4s^3 \sin t \cos t - s ds dt \\ &= \int_0^{2\pi} -\frac{4}{3}s^3 \cos t - s^4 \sin t \cos t - \frac{s^2}{2} \Big|_0^2 dt \\ &= \int_0^{2\pi} -\frac{32}{3} \cos t - 16 \sin t \cos t - 2 dt \\ &= \frac{32}{3} \sin t - 16 \sin^2 t - 2t \Big|_0^{2\pi} = -4\pi \end{aligned}$$

Calculate the flux of $\text{curl}(F)$ over the surface with outwards normals,



where $F = \langle z, 2xy, x+y \rangle$.

(The base is the unit circle.)

By Stokes' Theorem, $\iint_S \text{curl}(F) \cdot d\vec{S} =$

$$\begin{aligned} \oint_{\partial S} F \cdot d\vec{r} &= \int_0^{2\pi} \langle 0, 2\cos\theta\sin\theta, \cos\theta + \sin\theta \rangle \cdot \langle -\sin\theta, \cos\theta, 0 \rangle d\theta \\ &= \int_0^{2\pi} 0 + 2\cos^2\theta\sin\theta + 0 d\theta = \int_0^{2\pi} 2\cos^2\theta\sin\theta d\theta \\ &= -\frac{2}{3}\cos^3\theta \Big|_0^{2\pi} = 0. \end{aligned}$$

Thus we can solve this problem without even needing to parametrize this region.

Even more generally, if C is a simple closed curve, and S and T are two surfaces with $\partial S = \partial T = C$, then

$$\iint_S \text{curl}(F) \cdot d\vec{S} = \oint_C F \cdot d\vec{r} = \iint_T \text{curl}(F) \cdot d\vec{S}$$

by Stokes' Theorem. In particular, if

$$G = \text{curl}(F), \text{ then } \iint_S G \cdot d\vec{S} = \iint_T G \cdot d\vec{S} \text{ so}$$

long as $\partial S = \partial T$.

If G is a curl vector field, (in particular if its domain is simply connected and its divergence is 0), then its flux depends only on the boundary of the surface.

Ex: Find the flux of $F = \langle y^2 + e^{z^2}, \sin(x^3 + e^y) \rangle$ through the sphere of radius $\sqrt{x^2 + y^2 + z^2} = 3$ centered at $(10, 8, 0)$

Sol: $\text{div}(F) = 0$ and the domain is simply connected, so $\iint_S F \cdot d\vec{S} = 0$, as there is G such that $F = \text{curl}(G)$ and $\iint_S F \cdot d\vec{S} = \iint_S G \cdot d\vec{r}$, but ∂S is empty.

Ex: Let F be a constant vector field and S a closed surface. Then $\iint_S F \cdot d\vec{S} = 0$.

Sol: Let S be the cylinder of radius 2 and height 4, $0 \leq z \leq 4$, with a top but no bottom, oriented outwards.

What is the flux of $F = \langle y^2, z, 3x + 2 \rangle$ through S ?