

Divergence Theorem

Stokes Theorem was the generalized form of the circulation form of Green's Theorem:

$$\oint_C \mathbf{F} \cdot d\vec{r} = \iint_D \operatorname{curl}(F) dA$$

$$\oint_{\partial S} \mathbf{F} \cdot d\vec{r} = \iint_S \operatorname{curl}(F) \cdot d\vec{S}$$

When $F = \operatorname{curl}(G)$ (in which case $\operatorname{div}(F) = 0$), and $\partial S = \emptyset$ we had $\oint_{\partial S} G \cdot d\vec{r} = \iint_S F \cdot d\vec{S} = 0$. If $\operatorname{div}(F) = 0$ but $F \neq \operatorname{curl}(G)$, is $\iint_S F \cdot d\vec{S}$ still zero for closed S ? The answer turns out to be yes.

Divergence Theorem

Let S be a piecewise smooth, closed surface with interior E . (We could denote $S = \partial E$.)

If S is positively oriented and F is continuously differentiable, then

$$\iint_S \hat{\mathbf{F}} \cdot d\vec{S} = \iiint_E \operatorname{div}(F) \cdot dV.$$

(Here, \iint_S denotes that S is closed, much like \oint_C) Compare this to the flux form of Green's Theorem:

$$\oint_C (\mathbf{F} \cdot d\vec{N}) ds = \iint_D \operatorname{div}(F) dA.$$

In both cases, we are measuring the flow of stuff across the boundary by adding up all of the flow from each point in the region.

Ex3 Let S be the cone $x^2 + y^2 = z^2$, $0 \leq z \leq 1$, with a top which is oriented outwards. Calculate the flux of $\mathbf{F} = \langle x-y, x+z, z-y \rangle$ through S .

Sol: By the divergence theorem, $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div}(\mathbf{F}) dV$. Thus $\operatorname{div}(\mathbf{F}) = 1 + 0 + 1 = 2$.

Then $\iiint_E \operatorname{div}(\mathbf{F}) dV$ is, in spherical coordinates,

$$\int_0^{2\pi} \int_0^{\pi/4} \int_0^1 r^2 \cos^2 \theta \sin \phi \, dr \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi/4} \frac{2}{3} r^3 \sin \theta \cos^2 \theta \, d\phi \, d\theta =$$

$$\int_0^{2\pi} \int_0^{\pi/4} \frac{2}{3} \frac{\sin \theta}{\cos^3 \phi} \, d\phi \, d\theta = \int_0^{2\pi} \frac{1}{3} \frac{1}{\cos^2 \phi} \Big|_0^{\pi/4} d\theta = \int_0^{2\pi} \frac{2}{3} - \frac{1}{3} d\theta = \int_0^{2\pi} \frac{1}{3} d\theta$$

$$= \frac{2\pi}{3}$$

We can double check this by computing flux directly. S can be decomposed into the cone and the disk of radius 1 at $z=1$. Then flux is the sum of

$$\begin{aligned} & \int_0^{2\pi} \int_0^1 \langle r \cos \theta, r \sin \theta, r \rangle \cdot \langle r \cos \theta, r \sin \theta, 1 - rs \sin \theta \rangle \langle 0, 0, r \rangle dr d\theta \\ &= \int_0^{2\pi} \int_0^1 r - r^2 \sin \theta dr d\theta = \int_0^{2\pi} \frac{r^2}{2} - \frac{r^3}{3} \sin \theta \Big|_0^1 d\theta = \int_0^{2\pi} \frac{1}{2} - \frac{\sin \theta}{3} d\theta \\ &= \pi \end{aligned}$$

and $R(r, \theta) = (r \cos \theta, r \sin \theta, r)$

$$\mathbf{T}_r = \langle \cos \theta, \sin \theta, 1 \rangle$$

$$\mathbf{T}_\theta = \langle -r \sin \theta, r \cos \theta, 0 \rangle$$

$$\mathbf{N} = \langle -r \cos \theta, -r \sin \theta, r \rangle$$

$$\int_0^{2\pi} \int_0^1 \langle r \cos \theta - r \sin \theta, r \cos \theta + r, r - r \sin \theta \rangle \cdot \langle r \cos \theta, r \sin \theta, -r \rangle dr d\theta =$$
 ~~$\int_0^{2\pi} \int_0^1 r^2 \sin \theta - r^2 \cos \theta + r^2 \sin \theta - r^2 \cos \theta \, dr d\theta = \int_0^{2\pi} \int_0^1 0 \, dr d\theta = 0$~~

$$S_0^{2\pi} S_0^1 r^2 \cos^2 \theta - r^2 \cos \theta \sin \theta + r^2 \cos \theta \sin \theta + r^2 \sin^2 \theta = r^2 + r^2 \sin^2 \theta$$

$$= S_0^{2\pi} S_0^1 r^2 \cos^2 \theta + 2r^2 \sin^2 \theta - r^2 d\theta = S_0^{2\pi} \frac{\cos^2 \theta}{3} + \frac{2\sin^2 \theta}{3} - \frac{1}{3} d\theta =$$

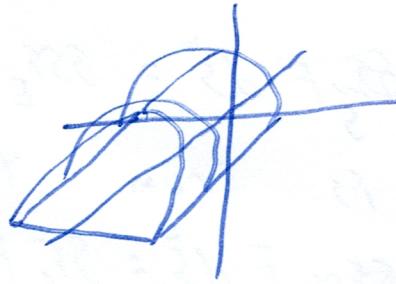
$$\left(\frac{\pi}{6} + \frac{\sin 2\theta}{12} \right) - \frac{2 \cos \theta}{3} - \frac{\theta}{3} \Big|_0^{2\pi} = \frac{\pi}{3} - \frac{2\pi}{3} = -\frac{\pi}{3}.$$

Thus the total flux is $\pi + (-\frac{\pi}{3}) = \frac{2\pi}{3}$.

Ex: Let S be the surface bounding the region given by $z \leq 1-x^2$, $z \geq 0$, $y \geq 0$, and $y+z \leq 2$. Let $F = \langle xy, x^2 + e^{x^2}, \sin(xy) \rangle$. Calculate $\iint_S F \cdot d\vec{s}$.

Sol: We could split this up

into three smooth pieces to compute the flux directly, but instead we apply the divergence theorem.



E can be described by the bounds of integration $-1 \leq x \leq 1$, $0 \leq z \leq 1-x^2$, $0 \leq y \leq 2-z$. Then the flux is equal to

$$S_{-1}^1 S_0^{1-x^2} S_0^{2-z} 3y \, dy \, dz \, dx \quad \text{since } \operatorname{div}(F) = y + 0 + 2y = 3y.$$

$$S_{-1}^1 S_0^{1-x^2} S_0^{2-z} 3y \, dy \, dz \, dx = S_{-1}^1 S_0^{1-x^2} 3 \frac{(2-z)^2}{2} \, dz \, dx =$$

$$S_{-1}^1 - \frac{3}{2} (2-z)^3 \Big|_0^{1-x^2} \, dx = S_{-1}^1 - \frac{1}{2} (x^2+1)^3 + 4 \, dx = S_{-1}^1 - \frac{x^6}{2} - \frac{3}{2} x^4 - \frac{3}{2} x^2 + 4x$$

$$= -\frac{x^7}{14} + \frac{3}{10} x^5 - \frac{x^3}{2} + \frac{7}{2} x \Big|_0^1 = -\frac{1}{14} + \frac{3}{10} - \frac{1}{2} + \frac{7}{2} = \underline{\underline{0.707}}$$

$$\frac{245 - 35 - 21 - 5}{75} = \frac{184}{75}$$