

Just like with Green's Theorem, we can apply the divergence theorem in cases where the surface is not closed.

Ex: Let  $\mathbf{F} = \langle y^2, z^2, x^2 \rangle$ . Compute the flux of  $\mathbf{F}$  through the upper hemisphere  $S$  of the unit sphere, oriented upwards.

Sol: The upper hemisphere plus the unit disk  $D$  is a closed surface, i.e. the new surface  $\tilde{S}$

$$\oint_S \mathbf{F} \cdot d\tilde{s} = \iiint_E \operatorname{div} \mathbf{F} dV = \iiint_E 0 dV = 0$$

This is from the divergence theorem. But,

$$\oint_{\tilde{S}} \mathbf{F} \cdot d\tilde{s} = \iint_S \mathbf{F} \cdot d\tilde{s} + \iint_D \mathbf{F} \cdot d\tilde{s}.$$

$$\begin{aligned} \text{This rearranges to } \iint_S \mathbf{F} \cdot d\tilde{s} &= \oint_S \mathbf{F} \cdot d\tilde{s} - \iint_D \mathbf{F} \cdot d\tilde{s} \\ &= - \iint_D \mathbf{F} \cdot d\tilde{s}. \end{aligned}$$

$$\begin{aligned} \text{This is } \iint_0^{2\pi} &\langle r^2 \sin^2 \theta, 0, r^2 \cos^2 \theta \rangle \cdot \langle 0, 0, -r \rangle dr d\theta \\ &= \int_0^{2\pi} \int_0^1 -r^3 \cos^2 \theta dr d\theta = \int_0^{2\pi} -\frac{r^4}{4} \cos^2 \theta \Big|_0^1 d\theta \\ &= \int_0^{2\pi} -\frac{\cos^2 \theta}{4} d\theta = -\frac{\pi}{4}. \end{aligned}$$

$$\text{Thus } \iint_S \mathbf{F} \cdot d\tilde{s} = \frac{\pi}{4}.$$

Ex: S given by  $x^2+y^2=9$  with height 4.  
Bottom, bat by no top.

Calculate  $\iint_S \mathbf{F} \cdot d\mathbf{S}$ , where  $\mathbf{F} = \langle 6x, 3y, z^2 \rangle$ .

Sol: Let T be the surface obtained  
by closing  $S_1$  i.e. adding a lid.

Then by the divergence theorem,

$$\iint_T \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} dV = \iiint_E 9+4z dV,$$

where E is the solid cylinder. Then  
this is

$$\int_0^{2\pi} \int_0^3 \int_0^4 (9+4z) r dz dr d\theta = \int_0^{2\pi} \int_0^3 (36r + 32r) dr d\theta =$$

$$\int_0^{2\pi} \int_0^3 68r dr d\theta = \int_0^{2\pi} 34r^2 \Big|_0^3 d\theta = \int_0^{2\pi} 306 d\theta =$$

$$72\pi.$$

But,  $\iint_T \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot d\mathbf{S} + \iint_D \mathbf{F} \cdot d\mathbf{S}$ , where D  
is the disk of radius 3 at height 4.

$$\iint_D \mathbf{F} \cdot d\mathbf{S} = \iint_0^{2\pi} \int_0^3 \langle 6r\cos\theta, 3r\sin\theta, 32r^2 \rangle \cdot \langle 0, 0, r \rangle dr d\theta =$$

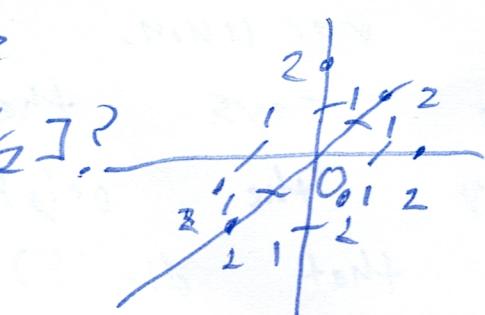
$$\int_0^{2\pi} \int_0^3 32r^3 dr d\theta = \int_0^{2\pi} 16r^2 \Big|_0^3 d\theta = \int_0^{2\pi} 144 d\theta = 288\pi.$$

Thus  $\iint_S \mathbf{F} \cdot d\mathbf{S} = 72\pi - 288\pi = 424\pi.$

A charge  $q$  at the origin of amount of exerts an electrostatic field  $F_q$  given by  $\frac{q}{4\pi\epsilon_0 r^3} \langle x, y, z \rangle$ , where  $\epsilon_0$  is an electrical constant representing the resistance in a vacuum. Then the divergence theorem says that, for surfaces not containing the origin,  $\iint_S F_q \cdot d\vec{s} = \iiint_E \operatorname{div}(F_q) dV$ . Notice that  $\operatorname{div}(F) = 0$ , so this is 0. If the surface does contain the origin, then the flux will be  $\frac{q}{\epsilon_0}$ . (We almost see this for the sphere in WT Q3.) Notice that the total charge inside is  $q$ , so this is the total charge divided by our constant. This is Gauss's Law.

The flux of the electrostatic field through a piecewise smooth closed surface is equal to  $\frac{Q}{\epsilon_0}$ , where  $Q$  is the total charge enclosed by the surface.

In other words, we can measure the total charge in an object by measuring the electrostatic field over its surface.

Ex: Suppose there is a charge placed  
 at  $(x, y, z)$  of amount  $x^2 + y^2 + z^2$   
 exactly when  $x, y$ , and  $z$  are integers.  
 What is the electrostatic  
 flux through the cube  
 $[-\frac{3}{2}, \frac{3}{2}] \times [-\frac{3}{2}, \frac{3}{2}] \times [-\frac{3}{2}, \frac{3}{2}]$ ? 

Sol: By Gauss's Law, the etc.  
 flux is the total charge divided by  $\epsilon_0$ .  
 The charge is found by "integrating"  
 the charge at every point far side the  
 cube, i.e. adding up all the charges  
 inside, i.e. all integer points with components  
 smaller than 2, i.e.  $(0, 0, 0), (1, 0, 0), (0, 1, 0),$   
 $(0, 0, 1), (1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 1)$ , plus the  
 points in the other quadrants. Adding up,  
 the total charge is  $0 + 6 \cdot 1 + 12 \cdot 2 + 8 \cdot 3 = 54$ .  
 Thus the flux is  $\frac{54}{\epsilon_0}$