MATH 146
Current Problems in Applied Mathematics: Dynamical Systems and Quantum Information

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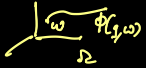
Section 1

Introduction

Ergodic theory

e.g. Hamiltonian mechanics
$\Omega = \mathbb{R}^6$ (positions, momenta in 3D)

Ergodic theory studies the statistical behavior of measurable actions of groups or semigroups on spaces.



Definition 1.1.
A left action, or flow, of a (semi)group G on a set Ω is a map $\Phi : G \times \Omega \to \Omega$ with the following properties:

evolution time

1. $\Phi(e, \omega) = \omega$, for the the identity element $e \in G$ and all $\omega \in \Omega$.
2. $\Phi(gh, \omega) = \Phi(g, \Phi(h, \omega))$, for all $g, h \in G$ and $\omega \in \Omega$.

"initial condition"

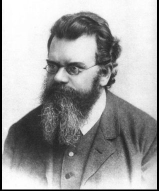
The set Ω is called the state space.

In this course, G will be an abelian group or semigroup that represents the time domain. Common choices include:

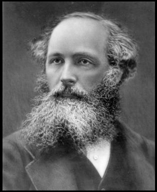
$$\mathbb{N}, \qquad \mathbb{Z}, \qquad \mathbb{R}_+, \qquad \mathbb{R}.$$

We write $\Phi^g \equiv \Phi(g, \cdot)$, $n \in \mathbb{N}, \mathbb{Z}$, and $t \in \mathbb{R}_+, \mathbb{R}$.

Ergodic theory



Ludwig Boltzmann



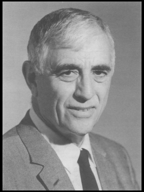
James Clerk Maxwell

Ergodic theory has its origin in the mid 19th century with the work of Boltzmann and Maxwell on statistical mechanics.

The term ergodic is an amalgamation of the Greek words ergo (έργο), which means *work*, and odos (οδός), which means *street*.

Ergodic theory

George David Birkhoff



Bernard Osgood Koopman

John von Neumann

The mathematical foundations of the subject were established by Koopman, von Neumann, Birkhoff, and many others, in work dating to the 1930s.

Modern ergodic theory is a highly diverse subject with connections to functional analysis, harmonic analysis, probability theory, topology, geometry, number theory, and other mathematical disciplines.

Observables and ergodic hypothesis

Rather than studying the flow Φ directly, ergodic theory focuses on its induced action on linear spaces of observables, e.g.,

$$\mathcal{F} = \{f : \Omega \to \mathcal{Y}\},$$

$\mathcal{F}$ is a linear space:
$f, g \in \mathcal{F}$ $h = f + g$ s.t. $h(\omega) = f(\omega) + g(\omega)$
$f \in \mathcal{F}$, c scalar $g = cf$ s.t. $g(\omega) = cf(\omega)$

for a vector space $\mathcal{Y}$ (oftentimes, $\mathcal{Y} = \mathbb{R}$ or $\mathbb{C}$).

Drawing on intuition from mechanical systems, Boltzmann postulated that time averages of observables should well-approximate expectation values with respect to a reference distribution, μ.

This is encapsulated in the ergodic hypothesis,

$$\underbrace{\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(\Phi^n(\omega))}_{\text{time average}} = \underbrace{\int_\Omega f \, d\mu}_{\text{space average}},$$

which is stipulated to hold for typical initial conditions $\omega \in \Omega$ and observables $f : \Omega \to \mathcal{Y}$ in a suitable class.

Operator-theoretic perspective

$f_1, f_2 \in \mathcal{F}: \; U^g(f_1+f_2) = (f_1+f_2)\circ\phi^g$
$= f_1\circ\phi^g + f_2\circ\phi^g$
$= U^g f_1 + U^g f_2$

$f \in \mathcal{F}$, c scalar: $U^g(cf) = (cf)\circ\phi^g = c(f\circ\phi^g)$
$= c\,U^g f$

Definition 1.2.

1. For every $g \in G$, the composition operator, or Koopman operator is the linear map $U^g : \mathcal{F} \to \mathcal{F}$ defined as
$$U^g f = f \circ \Phi^g. \quad \Leftrightarrow \quad (U^g f)(\omega) = f(\Phi^g(\omega))$$
2. The transfer operator $P^g : \mathcal{F}' \to \mathcal{F}'$ is the adjoint of U^g, defined as
$$P^g \nu = \nu \circ U^g.$$

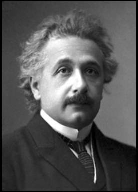
$\mathcal{F}'$ = space of linear functionals, i.e., linear functions $\nu : \mathcal{F} \to \mathbb{K}$

$\mathbb{K}$: scalar field ($\mathbb{R}$, $\mathbb{C}$)

Koopman and transfer operators allow the study of nonlinear dynamics using techniques from linear operator theory.

Quantum mechanics

Niels Bohr

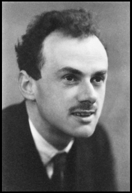


Albert Einstein

Max Planck

Quantum mechanics arose in the late 19th to early 20th century when it was realized that classical physics do not adequately describe phenomena such as the blackbody radiation spectrum, the photoelectric effect, and atomic spectral lines.

Quantum mechanics



Paul Dirac

Werner Heisenberg

Emmy Noether

Erwin Schrödinger

The mathematical formalism of quantum mechanics was developed by Schrödinger, Heisenberg, Dirac, Noether, von Neumann, and many others. The modern formulation of quantum mechanics makes heavy use of operator theory.

Dirac–von Neumann axioms of quantum mechanics

$H = L^2(\mathbb{R}^3)$ (hydrogen atom)

$H = \mathrm{span}\{|0\rangle, |1\rangle\}$ (spin system)

1. States are density operators, i.e., positive, trace-class operators $\rho : H \to H$ on a Hilbert space H, with $\operatorname{tr} \rho = 1$.

   $\rho \geq 0 \Leftrightarrow \forall f \in H$ $\langle f, \rho f \rangle \geq 0$

2. Observables are self-adjoint operators, $A : D(A) \to H$.

   ↳ subspace of H, domain of A

   $A^* = A$. If $A : H \to H$ is bounded, A^* is the unique operator s.t. $\langle f, Ag \rangle = \langle A^* f, g \rangle, \forall f, g \in H$

3. Measurement expectation and probability:

$$\mathbb{E}_\rho A = \operatorname{tr}(\rho A), \quad \mathbb{P}_\rho(\Omega) = \mathbb{E}_\rho(E(\Omega)), \quad A = \int_{\mathbb{R}} a \, dE(a).$$

e.g. For a self-adjoint matrix A, we have $A = \sum_i a_i E_i$

$a_i \in \mathbb{R}$ eigenvalue

E_i projection matrix onto the corresponding eigenspace

4. Unitary dynamics between measurements:

$$\rho_t = U^{t*} \rho_0 U^t.$$

$\operatorname{tr} \rho_t = \operatorname{tr}(U^{t*} \rho_0 U^t) = \operatorname{tr}(\rho_0 U^t U^{t*}) = \operatorname{tr} \rho_0 = 1$

$U^t : H \to H$ unitary

$U^t U^{t*} = U^{t*} U^t = I$

5. Projective measurement:

$$\rho|_e = \frac{\sqrt{e}\rho\sqrt{e}}{\operatorname{tr}(\sqrt{e}\rho\sqrt{e})}, \quad 0 < e \leq I.$$

$(A \leq B \Leftrightarrow B - A \geq 0)$

Classical probability density

Given measure μ on Ω we say that

$p \in L^1(\mu)$ $\left(\Leftrightarrow \int_\Omega |p| \, d\mu < \infty \right)$

is a probability density if

- $p \geq 0$
- $\int_\Omega p \, d\mu = 1$

Given classical observable $f \in L^\infty(\mu)$,

we define the expectation

$$\mathbb{E}_p f = \int_\Omega f p \, d\mu$$

Given a set S we can compute the probability of S wrt p as

$$\mathbb{E}_p \chi_S = \int_\Omega \chi_S \, p \, d\mu$$

$$\chi_S(\omega) = \begin{cases} 0 & \text{if } \omega \notin S \\ 1 & \text{if } \omega \in S \end{cases}$$

Quantum density operator

$\rho \in B_1(H)$

trace-class operators on H

- $\rho \geq 0$
- $\operatorname{tr} \rho = 1$

Given $A \in B(H)$

$$\mathbb{E}_\rho A = \operatorname{tr}(\rho A)$$

$$\mathbb{E}_\rho (E(S)) = \operatorname{tr}(E(S) \rho)$$

spectral measure

Interpretation of quantum mechanics

A vast subject in its own right, the interpretation of quantum mechanics can be approached by asking the following question:

- *Does quantum mechanics describe the world, or an observer's knowledge of the world?*

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- *Does quantum mechanics describe the world, or an observer's knowledge of the world?*

Quantum informational interpretations take the latter point of view.

- *Quantum information is the study of the information processing tasks that can be accomplished using quantum mechanical systems.*

(Nielsen & Chuang)

Connections with ergodic theory

Classical statistical	
Quantum mechanical	

$$
\begin{tikzcd}
P(\mu) \arrow[r, "P^t"] \arrow[d, "\Gamma"] & P(\mu) \arrow[d, "\Gamma"] \\
Q(H) \arrow[r, "\mathcal{P}^t"] & Q(H)
\end{tikzcd}
$$

$\Gamma : p \in L^1(\mu) \to \rho = \langle \sqrt{p}, \cdot \rangle \sqrt{p} \in B_1(H)$

For a measure-preserving flow $\Phi^t : \Omega \to \Omega$ on a probability space (Ω, Σ, μ):

$U^t f = f \circ \Phi^t$

- The Koopman operator $U^t : H \to H$ is unitary on $H = L^2(\mu)$ and thus defines a quantum system.
- The classical statistical dynamics on the space of probability densities with respect to μ, $P(\mu)$, induced by the transfer operator P^t consistently embeds into the quantum dynamics induced by U^t on the space of density operators $Q(H)$.

Classical

$f \in L^\infty(\mu)$

f is real-valued $\rightsquigarrow$

$p \in L^1(\mu)$ $\rightsquigarrow$

$\|p\|_{L^1(\mu)} = 1$

$$\mathbb{E}_p f = \mathbb{E}_\rho M_f$$

$$\int p f \, d\mu \qquad \operatorname{tr}(\rho M_f)$$

Quantum

$M_f \in B(H)$ $\quad H = L^2(\mu)$

$M_f g = fg$

multiplication operator

M_f is self-adjoint

ρ density operator

$\rho f = \langle \sqrt{p}, f \rangle \sqrt{p}$

projection operator along $\sqrt{p}$

"wavefunction"

$\|\sqrt{p}\|_H = 1$

Further reading

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[4] B. O. Koopman, "Hamiltonian systems and transformation in Hilbert space," *Proc. Natl. Acad. Sci.*, vol. 17, no. 5, pp. 315–318, 1931. DOI: 10.1073/pnas.17.5.315.

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