Section 2

Measure-preserving transformations; Ergodic theorems

MEASURE THEORY

A collection $\mathcal{A}$ of subsets of Ω is called an <u>algebra</u> if:

- $\emptyset \in \mathcal{A}$, $\Omega \in \mathcal{A}$
- $S \in \mathcal{A} \Rightarrow S^c \in \mathcal{A}$
- $S, T \in \mathcal{A} \Rightarrow S \cap T \in \mathcal{A}$, $S \cup T \in \mathcal{A}$

A <u>σ-algebra</u> Σ is a collection of subsets of Ω that is an algebra and moreover for any countable collection $\{A_I\} \in \Sigma$ then $\bigcup_I A_I \in \Sigma$

- Given any collection C of subsets of Ω, $\Sigma(C)$ is the smallest σ-algebra that contains C (Σ-algebra generated by C).

<u>Example</u>: If Ω is a topological space the <u>Borel σ-algebra</u> is the σ-algebra generated by the open sets in Ω.

A measure μ on a measurable space (Ω, Σ) is a map $\mu : \Sigma \to [0, \infty]$ such that: (i) $\mu(\emptyset) = 0$, (ii) for any disjoint countable collection $\{S_0, S_1, \dots \in \Sigma\}$, $\mu(\bigcup S_i) = \sum_i \mu(S_i)$.

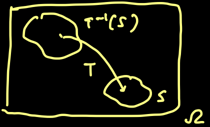
<u>Example</u>: Given a point $x \in \Omega$, define $\delta_x : \Sigma \to [0, 1]$ s.t. $\delta_x(S) = \chi_S(x)$ where $\chi_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases}$

A map $T:(\Omega_1, \Sigma_1) \to (\Omega_2, \Sigma_2)$ between measurable spaces is said to be <u>measurable</u> if $\forall S \in \Sigma_2$, $T^{-1}(S) \in \Sigma_1$.

Given a measurable map $T:(\Omega_1, \Sigma_1) \to (\Omega_2, \Sigma_2)$ and a measure $\mu : \Sigma_1 \to [0,\infty]$ we define the <u>pushforward</u> measure $T_* \mu : \Sigma_2 \to [0, \infty]$ as $T_*\mu(S) = \mu(T^{-1}(S))$

Check: For any $x \in \Omega_1$ $T_*(\delta_x) = \delta_{T(x)}$

Measure-preserving dynamical systems



Definition 2.1.

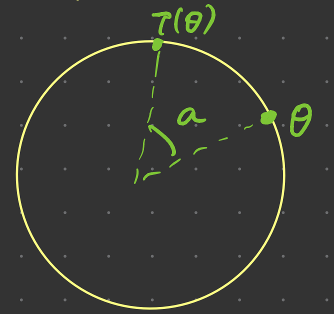
Let (Ω, Σ, μ) be a measure space.

1. A measurable map $T : \Omega \to \Omega$ is said to be measure-preserving if $T_*\mu = \mu$, i.e.,
$$\mu(T^{-1}(S)) = \mu(S), \quad \forall S \in \Sigma.$$
Conversely, we say that μ is an invariant measure for T.
2. A measure-preserving map $T : \Omega \to \Omega$ is said to be invertible measure-preserving if T is bijective and T^{-1} is also measure-preserving.
3. A measurable action $\Phi : G \times \Omega \to \Omega$ is μ-preserving if $\Phi^g : \Omega \to \Omega$ is μ-preserving for every $g \in G$.

Examples

(1) Circle rotation (discrete time, $G = \mathbb{Z}$)

$T : S^1 \longrightarrow S^1, \quad T(\theta) = \theta + a \mod 2\pi$



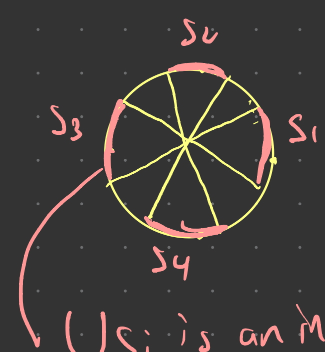
For any $a \in \mathbb{R}$, T preserves the Lebesgue (arclength) measure on the Borel σ-algebra on S^1.

(a) a is a rational multiple of 2π:

- The orbit of θ, i.e. the set $\{\theta, T(\theta), T^2(\theta), \dots\}$ is periodic.
- In this case, T preserves the discrete measure $\mu = \sum_{j=0}^{p} \delta_{\theta_j}$ where $\{\theta_0, \theta_1, \theta_2, \dots, \theta_p\}$ is a periodic orbit under T.

(b) a is an irrational multiple of 2π, the Lebesgue measure is the only Borel invariant measure under T.

(a) is not ergodic wrt Lebesgue measure:
e.g. $a = \pi/4$



$\bigcup s_i = S$ is an invariant set with $0 < \mu(S) < 1$

normalized Lebesgue measure

(2) Doubling map $T: S^1 \to S^1$

$$T(\theta) = 2\theta \mod 2\pi$$

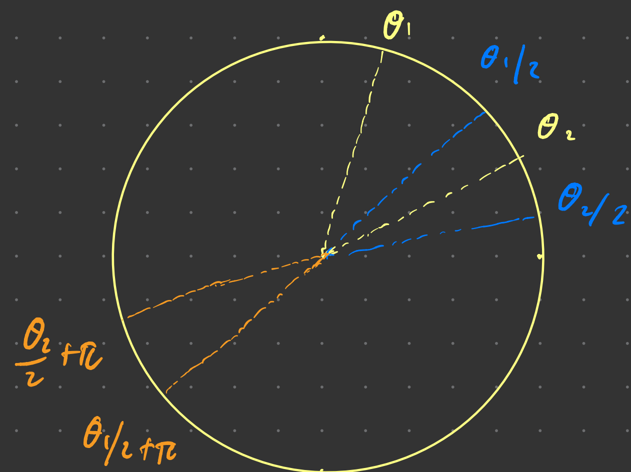
T preserves the Lebesgue measure:

It is enough to consider an interval $I = [\theta_1, \theta_2]$

$$\mu(I) = \theta_2 - \theta_1 \qquad T^{-1}(I) = \left[\frac{\theta_1}{2}, \frac{\theta_2}{2}\right] \cup \left[\frac{\theta_1}{2}+\pi, \frac{\theta_2}{2}+\pi\right]$$

$$\mu(T^{-1}(I)) = \mu\left(\left[\frac{\theta_1}{2}, \frac{\theta_2}{2}\right]\right) + \mu\left(\left[\frac{\theta_1}{2}+\pi, \frac{\theta_2}{2}+\pi\right]\right) = \frac{1}{2}\mu(I) + \frac{1}{2}\mu(I)$$

$$= \mu(I).$$



Lemma: If a σ-algebra Σ is generated by an algebra $\mathcal{A}$, T preserves a measure $\mu: \Sigma \to [0, \infty]$ iff it preserves it on the elements of $\mathcal{A}$.

For the Borel σ-algebra on S^1, we have that Σ is generated by the algebra $\mathcal{A}$ consisting of finite unions of intervals $[I_1, I_2]$.

As a result, T is measure-preserving iff $\mu(T^{-1}(I)) = \mu(I)$ for any interval I.

(3) Continuous-time flow on $\mathbb{R}^d$

$\dot{\vec{x}}(t) = \vec{v}(\vec{x}(t))$, $\vec{x}(t) \in \mathbb{R}^d$, $\vec{v}: \mathbb{R}^d \to \mathbb{R}^d$ vector field

$\phi^t : \mathbb{R}^d \to \mathbb{R}^d$ solution map associated with the initial-value poblem associated with $\vec{v}$, i.e., (under appropriate regularity assumptions)

$$\phi^t(\vec{x}) = \vec{x}(t) \quad \text{where} \quad \dot{\vec{x}}(t) = \vec{v}(\vec{x}(t)), \quad x(0) = x.$$

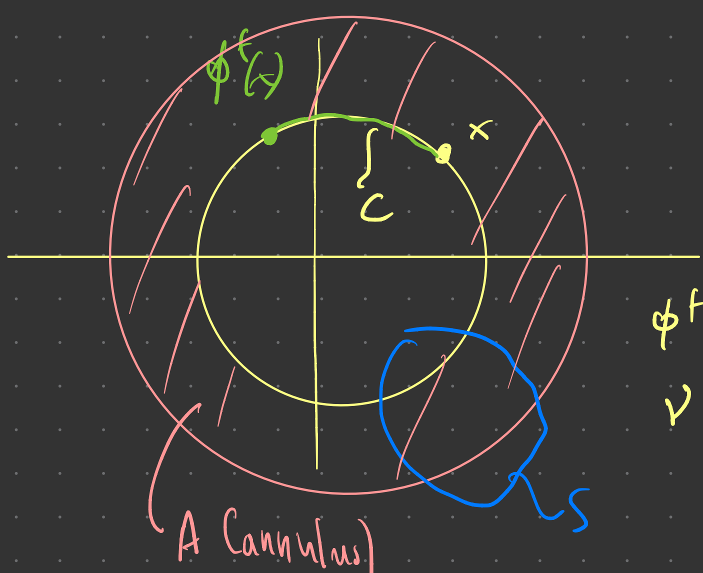
If $\operatorname{div} \vec{v} = \sum_{i=1}^{d} \frac{\partial v_i}{\partial x_i} = 0$ then ϕ^t preserves the Lebesgue measure on $\mathbb{R}^d$.

Example (simple harmonic oscillation): $\ddot{x}(t) = -\omega^2 x(t)$, $x(t) \in \mathbb{R}$

$$\begin{cases} \dot{x}(t) = \omega y(t) \\ \dot{y}(t) = -\omega x(t) \end{cases} \quad (*)$$

Consider the flow $\phi^t : \mathbb{R}^2 \to \mathbb{R}^2$ generated by $(*)$, vector field $\vec{v}(\vec{x}) = \begin{pmatrix} \omega y \\ -\omega x \end{pmatrix}$

$\vec{x} = (x_0, y_0)$



$$\phi^t(\vec{x}) = \operatorname{Re}\left((x_0 + i y_0) e^{i\omega t}\right)$$

ϕ^t preserves the Lebesgue measure since $\operatorname{div} v = 0$

$\nu(S) = \mu(S \cap A)$ is an inv. measure (non-ergodic)

$\tilde{\nu}(S) = \mu_C(S \cap C)$ is another invariant measure (ergodic)

↳ 1D Lebesgue measure on C

Recurrence

the set of points in S for which this property does not hold is included in a measurable set $N \in \Sigma$ s.t. $\mu(N) = 0$.

Theorem 2.2 (Poincaré).
Let $T : \Omega \to \Omega$ be a measure-preserving transformation of the probability space (Ω, Σ, μ). Let $S \in \Sigma$ be a measurable set with $\mu(S) > 0$. Then, under iteration by T, almost every point of S returns to S infinitely often. That is, for μ-a.e. $\omega \in S$, there exists a sequence $n_1 < n_2 < n_3 < \cdots$ of natural numbers, increasing to infinity, such that $T^{n_j}(\omega) \in S$ for all j.

Proof. For $N \geq 0$, let $S_N = \bigcup_{n=N}^{\infty} T^{-n}(S)$

↳ Set of points that return to S after iteration by $n \geq N$ steps.

Then, $\bigcap_{N=0}^{\infty} S_N$ is the set of points that return to S infinitely often.

Let $A = S \cap \left(\bigcap_{N=0}^{\infty} S_N\right) = \bigcap_{N=0}^{\infty} S_N$. (since $S \subseteq S_0$)

↳ Set of points in S that return to S infinitely often.

For each $\omega \in A$, there exist $n_1 < n_2 < n_3 < \ldots$ s.t. $T^{n_j}(\omega) \in S$. It is enough to show $\mu(A) = \mu(S)$.

Indeed, we have $T^{-1}(S_N) = S_{N+1}$, and since T is μ-preserving, $\mu(S_{N+1}) = \mu(T^{-1}(S_N)) = \mu(S_N)$.

$\Rightarrow \mu(S_N) = \mu(S_0)$.

Moreover, we have $S_0 \supseteq S_1 \supseteq S_2 \cdots$, so $\mu(A) = \mu\left(\bigcap_{N=0}^{\infty} S_N\right) = \mu(S_0) \geq \mu(S)$ since $S \subseteq S_0$.

But $\mu(A) \leq S$ since $A \subseteq S$, so we conclude that $\mu(A) = \mu(S)$ ∎

Ergodicity

Definition 2.3.

$\mu(\Omega) = 1$

Let (Ω, Σ, μ) be a probability space.

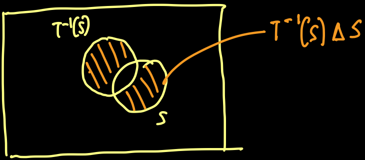
1. A measurable map $T : \Omega \to \Omega$ is said to be ergodic if for every T-invariant set, i.e., every $S \in \Sigma$ such that $T^{-1}(S) = S$ we have either $\mu(S) = 0$ or $\mu(S) = 1$.
2. A measurable action $\Phi : G \times \Omega \to \Omega$ is ergodic if for every $S \in \Sigma$ such that $\Phi^{-g}(S) = S$ for all $g \in G$ we have either $\mu(S) = 0$ or $\mu(S) = 1$.

1) Let $T : \Omega \to \Omega$ be any measurable map, let $x \in \Omega$ be any point. Then T is ergodic for the Dirac measure $\mu = \delta_x$.

Measure-theoretic characterization of ergodicity

Theorem 2.4.
Let $T : \Omega \to \Omega$ be a measure-preserving transformation of the probability space (Ω, Σ, μ). Then, the following are equivalent.

1. *T is ergodic.*
2. *The only measurable sets $S \in \Sigma$ such that $\mu(T^{-1}(S) \triangle S) = 0$ have either $\mu(S) = 0$ or $\mu(S) = 1$.*
3. *For every $S \in \Sigma$ with $\mu(S) > 0$, we have $\mu(\bigcup_{n=1}^{\infty} T^{-n}(S)) = 1$.*
4. *For every $R, S \in \Sigma$ with $\mu(R) > 0$ and $\mu(S) > 0$, there exists $n > 0$ with $\mu(T^{-n}(R) \cap S) > 0$.*



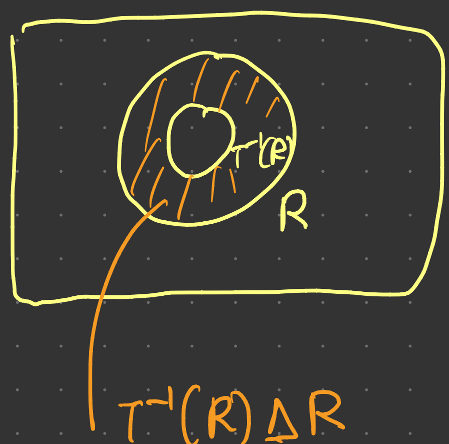
Orbits of sets of positive measure well-sample the invariant measure

Proof that (ii) $\Rightarrow$ (iii)

Let $S \in \Sigma$ have $\mu(S) > 0$. To show: $R = \bigcup_{n=1}^{\infty} T^{-n}(S)$ has $\mu(R) = 1$.

We have $T^{-1}(R) = \bigcup_{n=2}^{\infty} T^{-n}(S) \subseteq R$

Since μ is invariant, $\mu(T^{-1}(R)) = \mu(R)$



$\Rightarrow \mu(T^{-1}(R) \Delta R) = 0$

$\underset{(ii)}{\Longrightarrow}$ Either $\mu(R) = 0$ or $\mu(R) = 1$

However, $\mu(R) \geq \mu(T^{-1}(S)) = \mu(S) > 0$

$\Rightarrow \mu(R) = 1$. $\square$

Measure-theoretic characterization of ergodicity

Theorem 2.5.
Let (Ω, Σ, μ) *be a probability space.*

1. *A measure-preserving action* $\Phi : \mathbb{N} \times \Omega \to \Omega$ *is ergodic iff*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu(\Phi^{-n}(R) \cap S) = \mu(R)\mu(S), \quad \forall R, S \in \Sigma.$$

2. *A measure-preserving action* $\Phi : \mathbb{R}_+ \times \Omega \to \Omega$ *is ergodic iff*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mu(\Phi^{-t}(R) \cap S)\, dt = \mu(R)\mu(S), \quad \forall R, S \in \Sigma.$$

<u>Interpretation:</u> Under action of the dynamics, events represented by R and S become independent in a time-averaged sense.

Koopman operators on L^p spaces

Definition 2.6.
A measurable map $T : \Omega \to \Omega$ on a measure space (Ω, Σ, μ) is said to be nonsingular if it preserves null sets, i.e., if whenever $\mu(S) = 0$ we have $T_*\mu(S) = \mu(T^{-1}(S)) = 0$.

Notation.

- $\mathbb{L}(\Sigma) = \{f : \Omega \to \mathbb{R} : f \text{ is } \Sigma\text{-measurable}\}$.
- $L(\mu) = \{[f]_\mu : f \in \mathbb{L}(\Sigma)\}$.
- $L^p(\mu) = \{[f]_\mu \in L(\mu) : \int_\Omega |f|^p \, d\mu < \infty\}$.
- $L^\infty(\mu) = \{[f]_\mu \in L(\mu) : \operatorname{esssup}_\mu |f| < \infty\}$.

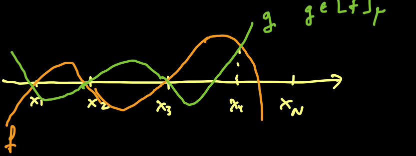
Banach spaces equipped with the norm $\|f\|_{L^p(\mu)} = \left(\int_\Omega |f|^p \, d\mu\right)^{1/p}$

$\|f\|_{L^\infty(\mu)} = \lim_{p \to \infty} \|f\|_{L^p(\mu)}$ (when μ is a finite measure)

$[f]_\mu = \{ g \in \mathbb{L}(\Sigma) \text{ s.t. } f = g \ \mu\text{-a.e.} \}$

Example: $\Omega = \mathbb{R}$, $\mu = \delta_{x_1} + \dots + \delta_{x_N}$

$[f]_\mu = \{$functions $g : \Omega \to \mathbb{R}$ s.t. $f(x_n) = g(x_n)$ for all $n \in \{1, \dots, N\}\}$

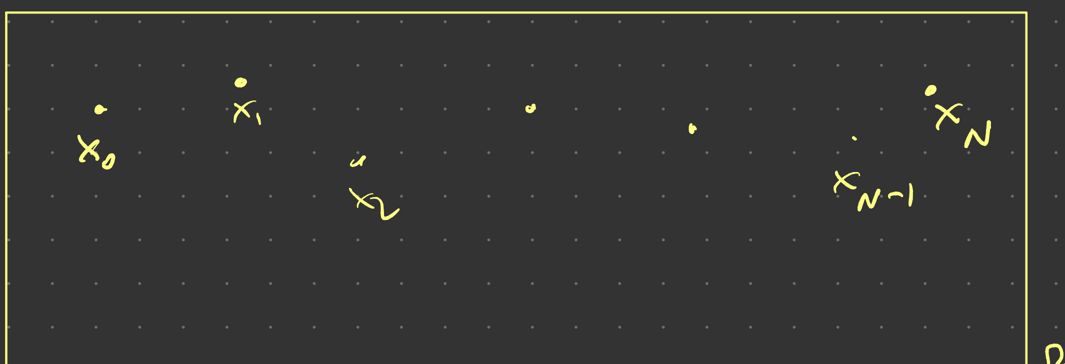


(i) If $T:\Omega \to \Omega$ is measurable, then the Koopman operator $U: f \longmapsto f \circ T$ maps $\mathbb{L}(\Sigma)$ into itself.

(ii) If $T:\Omega \to \Omega$ is non-singular wrt to a measure μ, then for any $f, g \in \mathbb{L}(\Sigma)$ if $[f]_\mu = [g]_\mu$ then $[f\circ T]_\mu = [g \circ T]_\mu$.

As a result, we can define a Koopman operator as a map $U: L(\mu) \to L(\mu)$ s.t. $U[f]_\mu = [f \circ T]_\mu$.

Non well-definition of Koopman operators for singular maps:



$$\mu_N = \frac{1}{N} \sum_{n=0}^{N-1} \delta_{x_n}$$

Let $X_N = \{x_0, \dots, x_{N-1}\}$ and suppose that T does not map X_N into itself. In particular assume $x_N = T(x_{N-1}) \notin X_N$

Then, $\mu_N(\{x_N\}) = 0$ but $\mu_N(T^{-1}\{x_N\}) = 1/N \Rightarrow T$ is singular wrt μ_N.

Let $f:\Omega \to \mathbb{R}$, $g:\Omega \to \mathbb{R}$ be such that $f(x_n) = g(x_n)$ for $n \in \{0, \dots, N-1\}$ and $f(x_N) \neq g(x_N)$. Then $[f]_{\mu_N} = [g]_{\mu_N}$ but $[f\circ T]_{\mu_N} \neq [g \circ T]_{\mu_N}$ (since $(f\circ T)(x_{N-1}) = f(x_N) \neq g(x_N) = (g \circ T)(x_{N-1})$. $\Rightarrow$ Koopman operator is not well defined on $[\ ]_{\mu_N}$ equiv. classes.

Koopman operators on L^p spaces

Proposition 2.7.

With notation as above, the following hold.

1. *If T is measurable, then the composition map $U : f \mapsto f \circ T$ maps $\mathbb{L}(\Sigma)$ into itself.*
2. *If T is nonsingular, then $\mathcal{U} : L(\mu) \to L(\mu)$ with $\mathcal{U}[f]_\mu = [Uf]_\mu$ is a well-defined linear map.*
3. *If T is nonsingular, then $L^\infty(\mu)$ is invariant under $\mathcal{U}$, i.e.,*

$$\mathcal{U} L^\infty(\mu) \subseteq L^\infty(\mu).$$

4. *If T is measure-preserving, then $\mathcal{U}$ is an isometry of $L^p(\mu)$, $1 \leq p \leq \infty$, i.e.,*

$$\|\mathcal{U}[f]_\mu\|_{L^p(\mu)} = \|[f]_\mu\|_{L^p(\mu)}.$$

5. *If T is invertible measure-preserving, then $\mathcal{U}$ is an isomorphism of $L^p(\mu)$, $1 \leq p \leq \infty$, i.e., $\mathcal{U}$ and $\mathcal{U}^{-1}$ are both isometries.*

Henceforth, we abbreviate $[f]_\mu \equiv f$, $U \equiv \mathcal{U}$.

Koopman operators on L^2

Hilbert space with inner product $\langle f_1, f_2 \rangle_{L^2(\mu)}$
and corresponding norm $\|f\|_{L^2(\mu)} = \sqrt{\langle f, f \rangle_{L^2(\mu)}}$

Notation.

- $\langle f_1, f_2 \rangle_{L^2(\mu)} = \int_\Omega f_1^* f_2 \, d\mu$.

The Koopman operator induced by a μ-preserving map $T : \Omega \to \Omega$ preservers Hilbert space inner products,

$$\langle Uf_1, Uf_2 \rangle_{L^2(\mu)} = \langle f_1, f_2 \rangle_{L^2(\mu)}.$$

If, in addition, T is invertible measure-preserving, then U is a unitary operator,

$$U^* = U^{-1}.$$

will allow us to define
a quantum system on $L^2(\mu)$.

Duality of L^p spaces

Notation.
For a probability space (Ω, Σ, μ), we let:

- $M_q(\Omega, \mu) = \left\{ \text{measures } \nu \ll \mu \text{ with density } \frac{d\nu}{d\mu} \in L^q(\mu) \right\}$.
- Duality pairing: $\langle \cdot, \cdot \rangle_\mu : L^p(\mu)^* \times L^p(\mu) \to \mathbb{R}$, $\langle \alpha, f \rangle_\mu = \alpha f$.

For $1 \leq p < \infty$, we can identify functionals in $L^p(\mu)^*$ with measures in $M_q(\Omega, \mu)$, $\frac{1}{p} + \frac{1}{q} = 1$, through the map $\iota_q : M_q(\Omega, \mu) \to L^p(\mu)^*$,

$$(\iota_q \nu) f = \int_\Omega f \rho \, d\mu, \quad \rho = \frac{d\nu}{d\mu}.$$

Equipping $M_q(\Omega, \mu)$ with the norm

$$\|\nu\|_{M_q(\Omega,\mu)} = \left\| \frac{d\nu}{d\mu} \right\|_{L^q(\mu)},$$

ι_q becomes an isomorphism of Banach spaces. Thus, we have

$$L^p(\mu)^* \simeq M_q(\Omega, \mu) \simeq L^q(\mu), \quad 1 \leq p < \infty, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

DUAL SPACES

- Let $(\mathcal{F}, \|\cdot\|_{\mathcal{F}})$ be a normed space. The continuous dual, $\mathcal{F}^*$ of $\mathcal{F}$ is the set of bounded linear functionals $\alpha : \mathcal{F} \to \mathbb{C}$

$$\sup_{f \in \mathcal{F}\setminus\{0\}} \frac{|\alpha f|}{\|f\|_{\mathcal{F}}} < \infty .$$

$\mathcal{F}^*$ is a Banach space equipped with the norm

$$\|\alpha\|_{\mathcal{F}^*} = \sup_{f \in \mathcal{F}\setminus\{0\}} \frac{|\alpha f|}{\|f\|_{\mathcal{F}}}$$

SIGNED / COMPLEX MEASURES (on measurable space (Ω, Σ))

- Signed measure: $\mu : \Sigma \to \mathbb{R}$, $\mu(S) = \mu_+(S) - \mu_-(S)$ where $\mu_+, \mu_- : \Sigma \longrightarrow [0, \infty]$ are measures

- Complex measure $\mu : \Sigma \to \mathbb{R}$, $\mu(S) = \mu_r(S) + i\mu_i(S)$ where $\mu_r, \mu_i : \Sigma \longrightarrow \mathbb{R}$ are signed measures

ABSOLUTE CONTINUITY

We say that a measure ν on (Ω, Σ) is absolutely continuous with respect to a measure μ on (Ω, Σ), denoted as $\nu \ll \mu$ if for every set $S \in \Sigma$ such that $\mu(S) = 0$ we have $\nu(S) = 0$.

Radon Nikodym thm: If $\nu \ll \mu$ there exists a unique element $\rho \in L^1(\mu)$ such that for every $S \in \Sigma$, $\nu(S) = \int_S \rho \, d\mu$. We typically write $\rho = \frac{d\nu}{d\mu}$

Conversely, given $\rho \in L^1(\mu)$, we can define a measure ν s.t. $\nu(S) = \int_S \rho \, d\mu$ and ν is a.c. wrt. μ.

Examples
- μ = Lebesgue measure on $\mathbb{R}$, $\rho \in L^1(\mu)$ $\rho(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-x^2/\sigma^2}$
  $\rightsquigarrow$ ν is a Gaussian prob. measure on $\mathbb{R}$, and $\nu \ll \mu$.

- μ = Lebesgue measure on $\mathbb{R}$, $\nu = \delta_x$ at some $x \in \mathbb{R}$. In this case ν is not absolutely continuous wrt μ (because $\mu(\{x\}) = 0$ but $\nu(\{x\}) = 1$).

RIESZ REPRESENTATION THEOREM

- If $\mathcal{F}$ is a Hilbert space, then for every $\alpha \in \mathcal{F}^*$ there exists a unique $a \in \mathcal{F}$ such that $\alpha f = \langle a, f \rangle_{\mathcal{F}}$ for every $f \in \mathcal{F}$. Conversely every $a \in \mathcal{F}$ induces a functional $\alpha \in \mathcal{F}^*$ s.t. $\alpha f = \langle a, f \rangle_{\mathcal{F}}$.

Transfer operators on L^p

Definition 2.8.
With the notation of Proposition 2.7, the transfer operator $P : L^1(\mu) \to L^1(\mu)$ is is the unique operator satisfying

$$\int_S Pf \, d\mu = \int_{T^{-1}(S)} f \, d\mu, \quad \forall f \in L^1(\mu).$$

We define $P : L^p(\mu) \to L^p(\mu)$, $1 < p \leq \infty$ by restriction of $P : L^1(\mu) \to L^1(\mu)$.

Proposition 2.9.
Under the identification $L^1(\mu)^* \simeq L^\infty(\mu)$, the transpose $P' : L^1(\mu)^* \to L^1(\mu)^*$ of the transfer operator $P : L^1(\mu) \to L^1(\mu)$ is identified with the Koopman operator $U : L^\infty(\mu) \to L^\infty(\mu)$; that is,

$$\int_\Omega f(Pg) \, d\mu = \int_\Omega (Uf) g \, d\mu, \quad \forall f \in L^\infty(\mu), \quad \forall g \in L^1(\mu).$$

Duality between Koopman and transfer operators

Proposition 2.10.

Let $1 \leq p < \infty$. Then, under the identification $L^p(\mu)^* \simeq L^q(\mu)$, $\frac{1}{p} + \frac{1}{q} = 1$, the following hold: $U'\alpha = \alpha \circ U$

1. *The transpose $U' : L^p(\mu)^* \to L^p(\mu)^*$ of the Koopman operator $U : L^p(\mu) \to L^p(\mu)$ is identified with the transfer operator $P : L^q(\mu) \to L^q(\mu)$; that is,*

$$\langle f, Ug \rangle_\mu = \langle Pf, g \rangle_\mu, \quad \forall f \in L^q(\mu), \quad \forall g \in L^p(\mu).$$

2. *The transpose $P' : L^p(\mu)^* \to L^p(\mu)^*$ of the transfer operator $P : L^p(\mu) \to L^p(\mu)$ is identified with the Koopman operator $U : L^q(\mu) \to L^q(\mu)$; that is,*

$$\langle f, Pg \rangle_\mu = \langle Uf, g \rangle_\mu, \quad \forall f \in L^q(\mu), \quad \forall g \in L^p(\mu).$$

Duality between Koopman and transfer operators

Corollary 2.11.

1. *For* $1 < p < \infty$*,* $U : L^p(\mu) \to L^p(\mu)$ *and* $P : L^p(\mu) \to L^p(\mu)$ *satisfy*

$$U = U'', \quad P = P''.$$

$\langle f, Ug \rangle_{L^2(\mu)} = \langle Pf, g$

2. *In the Hilbert space case,* $p = 2$*, we have* $P = U^*$*.*
3. *For* $1 \leq p \leq \infty$*,* P *has unit operator norm,* $\|P\|_{L^p(\mu)} = 1$*.*

Lemma 2.12.
With the notation of Proposition 2.8, if $T : \Omega \to \Omega$ *is invertible measure-preserving then* $P : L^p(\mu) \to L^p(\mu)$ *is the inverse of* $U : L^p(\mu) \to L^p(\mu)$*,* $P = U^{-1}$*.*

Spectral characterization of ergodicity

Observe that the Koopman operator $U : \mathcal{F} \to \mathcal{F}$ on any function space $\mathcal{F}$ has an eigenvalue equal to 1 with a constant corresponding eigenfunction, $\mathbb{1} : \Omega \to \mathbb{R}$,

$$U\mathbb{1} = \underbrace{\lambda}_{\text{eigenvalue } \lambda = 1.}\mathbb{1}, \quad \mathbb{1}(\omega) = 1.$$

Theorem 2.13.
Let $T : \Omega \to \Omega$ be a measure-preserving transformation of a probability space (Ω, Σ, μ). Then, μ is ergodic iff the eigenvalue equal to 1 of the associated Koopman operator U on $L(\mu)$ (and thus on any of the $L^p(\mu)$ spaces with $1 \leq p \leq \infty$) is simple, i.e.,

$$Uf = f \implies f = \text{const. } \mu\text{-a.e.}$$

Assume: The system is measure-preserving, ergodic. $f \in \mathbb{L}(\Sigma)$ $Uf = f$ μ-a.e.

To show: $f = \text{const.}$ μ-a.e.

Define $\Omega_{k,n} = f^{-1}\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)\right)$, $k \in \mathbb{Z}$, $n \in \mathbb{N}$

$$= \left\{ \omega \in \Omega : \frac{k}{2^n} \leq f(\omega) < \frac{k+1}{2^n} \right\}$$

$T^{-1}(\Omega_{k,n}) = \left\{ \omega \in \Omega : \frac{k}{2^n} \leq f(T(\omega)) < \frac{k+1}{2^n} \right\}$. $\quad S = \Omega_{k,n} \Delta T^{-1}(\Omega_{k,n})$

Then $S \subseteq \{ \omega \in \Omega : f(T(\omega)) \neq f(\omega) \} \Rightarrow \mu(S) = 0$.
$\quad$ ↳ since $Uf = f$ μ-a.e.

By ergodicity, either $\mu(\Omega_{k,n}) = 0$ or $\mu(\Omega_{k,n}) = 1$. For each n, $\Omega = \bigsqcup_{k \in \mathbb{Z}} \Omega_{k,n}$, so there is a unique k_n s.t. $\mu(\Omega_{k_n,n}) = 1$. Moreover, $\Omega_{k_{n+1}, n+1} \subseteq \Omega_{k_n, n}$.

Let $Q = \bigcap_{n=1}^{\infty} \Omega(k_n, n)$. On Q f is constant.

We have $\mu(Q) = \mu\left(\bigcap_{n=1}^{\infty} \Omega(k_n, n)\right) = \lim_{n \to \infty} \mu(\Omega_{k_n,n}) = 1 \Rightarrow f$ is constant on a set of full measure.

Application (Circle rotation)

Claim: $T: S^1 \to S^1$, $T(\theta) = \theta + a \mod 2\pi$ is ergodic wrt. Lebesgue measure iff a is an irrational multiple of 2π.

First, suppose that a is rational. Then, there exists $p \in \mathbb{Z}$ s.t. $e^{ipa} = 1$. Let $f(\theta) = e^{ip\theta}$. Then $Uf(\theta) = f(T(\theta)) = e^{ip(\theta+a)} = e^{ip\theta} = f(\theta)$.

$\Rightarrow$ f is an eigenfunction of U corresponding to e-value 1, but f is not constant μ-a.e.

$\Rightarrow$ T is not ergodic.

Conversely, suppose that $a/2\pi$ is irrational and $Uf = f$, $f \in L^2(\underbrace{\mu}_{\text{normalized Lebesgue}})$. Let $\phi_j(\theta) = e^{ij\theta}$ be the $\underbrace{\text{Fourier basis}}_{\langle \phi_j, \phi_k \rangle_{L^2(\mu)} = \delta_{jk}}$ of $L^2(\mu)$. Then $f = \sum_{j \in \mathbb{Z}} c_j \phi_j$, where

$c_j = \langle \phi_j, f \rangle_{L^2(\mu)} = \int_{S^1} e^{-ij\theta} f(\theta) \frac{d\theta}{2\pi}$. Observing that $U\phi_j(\theta) = e^{ij(\theta+a)} = e^{ija}\phi_j(\theta)$

(i.e., ϕ_j is an e-function of U of e-val. e^{ija}), we have $Uf = \sum_j c_j U\phi_j = \sum_j c_j e^{ija} \phi_j$

Thus, $Uf = f \Leftrightarrow Uf - f = 0 \Leftrightarrow \sum_{j \in \mathbb{Z}} c_j(e^{ija} - 1)\phi_j = 0 \Rightarrow c_j(e^{ija} - 1) = 0 \Rightarrow c_j = 0$ unless $j = 0$

since ϕ_j is an O-N basis

since a is irrational

$\Rightarrow f = c_0 \phi_0 \Rightarrow f$ is constant μ-a.e. $\Rightarrow T$ is ergodic with respect to Lebesgue measure. ∎

Spectral characterization of ergodicity

Theorem 2.14.

1. *Let* $\Phi : \mathbb{N} \times \Omega \to \Omega$ *be a measure-preserving action and* U^n*,* $n \in \mathbb{N}$*, the associated Koopman operators on any of* $L(\mu)$ *or* $L^p(\mu)$*,* $1 \leq p \leq \infty$*. Then* Φ *is ergodic iff* $U^n f = f$ *for all* $n \in \mathbb{N}$ *implies that* f *is constant* μ*-a.e.*
2. *Let* $\Phi : \mathbb{R}_+ \times \Omega \to \Omega$ *be a measure-preserving action and* U^t*,* $t \in \mathbb{R}_+$*, the associated Koopman operators on any of* $L(\mu)$ *or* $L^p(\mu)$*,* $1 \leq p \leq \infty$*. Then,* Φ *is ergodic iff* $U^t f = f$ *for all* $t \in \mathbb{R}_+$ *implies that* f *is constant* μ*-a.e.*

Pointwise ergodic theorem

Theorem 2.15 (Birkhoff).

Let $T : \Omega \to \Omega$ be a measure-preserving transformation of a probability space (Ω, Σ, μ) with associated Koopman operator $U : L^1(\mu) \to L^1(\mu)$. Then, for every $f \in L^1(\mu)$ and μ-a.e. $\omega \in \Omega$,

$$f_N(\omega) := \frac{1}{N} \sum_{n=0}^{N-1} f(T^n(\omega))$$

converges to a function $\bar{f} \in L^1(\mu)$ that satisfies

$$U\bar{f} = \bar{f}, \quad \int_\Omega f \, d\mu = \int_\Omega \bar{f} \, d\mu.$$

In particular, if T is ergodic, then for μ-a.e. $\omega \in \Omega$,

$$\bar{f}(\omega) = \int_\Omega f \, d\mu.$$

CONVERGENCE/TOPOLOGIES IN NORMED SPACES

Let $(F, \|\cdot\|)$ be a normed space.

- We say that a sequence $f_1, f_2, \ldots \in F$ converges to $f \in F$ if $\lim_{n\to\infty} \|f - f_n\| = 0$
- We say that a sequence $f_1, f_2, \ldots \in F$ converges to $f \in F$ weakly if for every $\alpha \in F^*$
$$\lim_{n\to\infty} (\alpha(f - f_n)) = 0$$

- Convergence in norm implies weak convergence but the converse is not true.

e.g. $F = \ell^2$, $f_n = (0, \ldots, 0, 1, 0, \ldots)$ ← n-th entry.

$\|(c_j)_j\|_{\ell^2} = \sqrt{\sum_j |c_j|^2}$

We have $\|f_n - f_m\|_{\ell^2} = \sqrt{2}$ for any $n, m \in \mathbb{N}$ $n \neq m$. Thus f_n does not converge in norm. However, it converges weakly to 0. Indeed, by the Riesz representation thm, for any $\alpha \in F^*$ we have $\alpha g = \langle a, g \rangle_{\ell^2}$ for some $a \in \ell^2$ and every $g \in \ell^2$. As a result, $\alpha f_n = a_n$, and since ($a = (a_1, a_2, \ldots)$)

$\|a\|_{\ell^2} = \sqrt{\sum_{n=1}^{\infty} |a_n|^2} < \infty$ we have $a_n \longrightarrow 0$ as $n \to \infty$. Thus, for every $\alpha \in F^*$,

$$\lim_{n\to\infty} \alpha f_n = 0 \implies f_n \xrightarrow[\text{weakly}]{} 0$$

CONVERGENCE/TOPOLOGY OF THE DUAL SPACE

Recall that F^* is a Banach space equipped with the norm

$$\|\alpha\|_{F^*} = \sup_{f \in F \setminus \{0\}} \frac{|\alpha f|}{\|f\|_F}$$

As a result F^* can be equipped with the corresponding norm and weak topologies.
In addition we have the weak-* topology of F^* induced from the weak topology of F. We say that a sequence $\alpha_1, \alpha_2, \ldots \in F^*$ converges to $\alpha \in F^*$ in weak-* sense if for every $f \in F$ $\lim_{n \to \infty} \alpha_n f = \alpha f$.

↳ The weak-* topology of F^* is the smallest topology that makes the maps $\alpha \in F^* \longmapsto \alpha f$ continuous for every $f \in F$.

Banach-Alaoglu theorem: The unit ball in F^* (i.e. $B_1(F^*) = \{\alpha \in F^* : \|\alpha\|_{F^*} = 1\}$) is weak-* compact.

CONVERGENCE/TOPOLOGIES OF OPERATORS

Let F_1 be a normed space and F_2 a Banach space. We say that a linear map $A: F_1 \to F_2$ is <u>bounded</u> if $\sup_{f \in F_1 \setminus \{0\}} \frac{\|Af\|_{F_2}}{\|f\|_{F_1}} < \infty$. The space of such bounded linear maps, $B(F_1, F_2)$ is a Banach space equipped with the operator norm

$$\|A\|_{B(F_1,F_2)} = \sup_{f \in F_1 \setminus \{0\}} \frac{\|Af\|_{F_2}}{\|f\|_{F_1}}.$$

We say that a sequence $A_1, A_2, \ldots \in B(F_1, F_2)$ converges to $A \in B(F_1, F_2)$ in

(i) <u>norm topology</u> if $\lim_{n\to\infty} \|A_n - A\| = 0$.

(ii) <u>strong operator topology</u> if $\forall f \in F_1$, $\lim_{n\to\infty} \|(A_n - A)f\|_{F_2} = 0$.

(iii) <u>weak operator topology</u> if $\forall f \in F_1$ and $\alpha \in F_2^*$ $\lim_{n\to\infty} \alpha(A_n f) = \alpha(Af)$.

norm converge $\Rightarrow$ strong convergence $\Rightarrow$ weak convergence, but in general the converse is not true.

Example $F_1 = F_2 = \ell^2$, $f_n = (0, \dots, 0, 1, 0, \dots)$ as before. Recall $\langle g, h \rangle_{\ell^2} = \sum_{n=0}^{\infty} g_n^* h_n$ where $g = (g_0, g_1, \dots)$, $h = (h_0, h_1, \dots)$. Let $\Pi_n \in B(\ell^2)$ be defined as $\Pi_n g = \langle f_n, g \rangle f_n = (0, \dots, 0, g_n, 0 \dots)$

↳ orthogonal projection along f_n

$\Pi_n^2 = \Pi_n$, $\Pi_n^* = \Pi_n$

(i) Π_n converges weakly to 0: Since ℓ^2 a Hilbert space, i.e. $\ell^2 \simeq \ell^2$, it is enough to show that for every $g, h \in \ell^2$, $\lim_{n\to\infty} \langle g, \Pi_n h \rangle = 0$. Indeed,

$\lim_{n\to\infty} \langle \underbrace{g}_{(g_0, g_1, \dots)}, \underbrace{\Pi_n h}_{(0, \dots, 0, h_n, 0, \dots)} \rangle = \lim_{n\to\infty} g_n^* h_n = 0$ since $g, h \in \ell^2$. Similarly $\lim_{n\to\infty} \|\Pi_n g\| = \lim_{n\to\infty} |g_n| = 0$, so $\Pi_n \to \Pi$ strongly. However, we have

$\sup_{g \in \ell^2 \setminus \{0\}} \frac{\|\Pi_n g\|}{\|g\|} = 1$ (choosing $g = f_n$), similarly $\sup_{g \in \ell^2 \setminus \{0\}} \frac{\|(\Pi_m - \Pi_n) g\|}{\|g\|} = 1$

whenever $m \neq n$, choosing $g = f_n$, so Π_n does <u>not converge</u> in operator norm.

Defining $\tilde{\Pi}_N = \sum_{n=0}^{N-1} \Pi_n$, we can similarly show that $\tilde{\Pi}_N \xrightarrow{s} I$ but it does not converge in operator norm.

↳ orthogonal projection onto $\mathrm{span}\{f_0, \dots, f_{N-1}\}$

Mean ergodic theorem

Theorem 2.16 (von Neumann).

Let $T : \Omega \to \Omega$ be a measure-preserving transformation of a probability space (Ω, Σ, μ) with associated Koopman operator $U : L^2(\mu) \to L^2(\mu)$. Let $\Pi : L^2(\mu) \to L^2(\mu)$ be the orthogonal projection onto the eigenspace of U corresponding to eigenvalue 1. Then, the sequence of operators $U_N = N^{-1} \sum_{n=0}^{N-1} U^n$ converges strongly to Π, i.e.,

$$\lim_{N \to \infty} U_N f = \Pi f, \quad \forall f \in L^2(\mu).$$

In particular, if T is ergodic, Π is the projection onto the 1-dimensional subspace of $L^2(\mu)$ containing μ-a.e. constant functions, i.e.,

$$\Pi f = \langle \mathbb{1}, f \rangle_{L^2(\mu)} \mathbb{1} = \left(\int_\Omega f \, d\mu \right) \mathbb{1}.$$

FINITE-RANK APPROXIMATIONS OF THE KOOPMAN OPERATOR

$T: \Omega \to \Omega$, measure-preserving with invariant probability measure μ.
$U: L^2(\mu) \to L^2(\mu)$, $Uf = f \circ T$, Koopman operator.

Given: $\{\phi_0, \phi_1, \dots, \}$ orthonormal basis of $L^2(\mu)$, i.e. $\forall f \in L^2(\mu)$
we have $f = \lim_{L \to \infty} f_L$ (in norm topology of $L^2(\mu)$) where $f_L = \sum_{l=0}^{L-1} \hat{f}_l \phi_l$
where $\hat{f}_l = \langle \phi_l, f \rangle_{L^2(\mu)}$.

Example: $\Omega = S^1$, μ normalized Lebesgue measure, an o-n basis of $L^2(\mu)$ is the Fourier basis, $\phi_l(\theta) = e^{il\theta}$ $\theta \in [0, 2\pi)$, $l \in \mathbb{Z}$. Given $f \in L^2(\mu)$, we have

$$\hat{f}_l = \langle \phi_l, f \rangle = \int_0^{2\pi} e^{-il\theta} f(\theta) \frac{d\theta}{2\pi}.$$

Define orthogonal projections $\Pi_L : L^2(\mu) \to L^2(\mu)$ such that $\operatorname{ran} \Pi_L = \operatorname{span}\{\phi_0, \dots, \phi_{L-1}\}$.
Explicitly $\Pi_L f = f_L \equiv \sum_{l=0}^{L-1} \hat{f}_l \phi_l$.

We have a family of projected Koopman operators $U_L : L^2(\mu) \to L^2(\mu)$ given by
$U_L = \Pi_L U \Pi_L$.

$U_L = \Pi_L U \Pi_L$ can be represented by an $L \times L$ matrix $A_L = [A_{ij}]_{i,j=0}^{L-1}$ where $A_{ij} = \langle \phi_i, U\phi_j \rangle$. In particular, given $f \in L^2(\mu)$ we have $g = U_L f$ where $g = (\hat{g}_0, \dots, \hat{g}_{L-1}, 0, \dots)$ and $\vec{g} = (\hat{g}_0, \dots, \hat{g}_{L-1})^T$ satisfies $\vec{g} = A_L \vec{f}$ with $\vec{f} = (\hat{f}_0, \dots, \hat{f}_{L-1})^T$.

<u>Example 1)</u>: $T: S^1 \to S^1$ circle rotation, $T(\theta) = \theta + a \mod 2\pi$.

We have $U\phi_j(\theta) = \phi_j(T(\theta)) = e^{ij(\theta+a)} = e^{ija} e^{ij\theta} = e^{ija}\phi_j(\theta)$

(In this case the ϕ_j are eigenvectors of U corresponding to eigenvalue e^{ija})

$A_{ij} = \langle \phi_i, U\phi_j \rangle = \langle \phi_i, \phi_j \rangle e^{ija} = \delta_{ij} e^{ija}$

$$\Rightarrow A_L = [A_{i,j}]_{i,j=-L}^{L} = \begin{pmatrix} e^{-iLa} & & & & & & 0 \\ & \ddots & & & & & \\ & & e^{-ia} & & & & \\ & & & 1 & & & \\ & & & & e^{ia} & & \\ & & & & & \ddots & \\ 0 & & & & & & e^{iLa} \end{pmatrix}$$

<u>Example 2</u>: $T: S^1 \to S^1$, doubling map, $T(\theta) = 2\theta \mod 2\pi$

$U\phi_j(\theta) = e^{i2j\theta} = \phi_{2j}(\theta) \Rightarrow \langle \phi_i, U\phi_j \rangle = \langle \phi_i, \phi_{2j} \rangle = \delta_{i,2j} = A_{ij}$

$$A_L = \begin{pmatrix} \ddots & & & & \\ & & 1 & & \\ & & 0 & & \\ & & 0 & 1 & 0 \\ & & & & 0 \\ & & & & 1 \\ & & & & & \ddots \end{pmatrix}$$

Proposition As $L \to \infty$, $U_L = \Pi_L U \Pi_L$ converges to U strongly.

Proof. First, observe that Π_L converges to I strongly since $\{\phi_\ell\}$ is an o-n basis of $L^2(\Gamma)$, i.e. $\Pi_L f \xrightarrow[L\to\infty]{} f$ for every $f \in L^2(\Gamma)$. As a result, since U is bounded, $\tilde{U}_L = U\Pi_L$ converges strongly to U. The strong convergence of $U_L = \Pi_L \tilde{U}_L$ to U will follow from the following lemma:

Lemma: Let A_L and B_L converge strongly to A and B, respectively. Assume that A_L is uniformly bounded i.e., $\sup_L \|A_L\| = a < \infty$. Then, $A_L B_L$ converges strongly to AB.

Of particular interest is the case where (G, τ_G) and (Ω, τ_Ω) are topological spaces and $\Phi : G \times \Omega \to \Omega$ is a continuous, and thus Borel-measurable, action. We let $\mathfrak{B}(\Omega)$ denote the Borel σ-algebra of Ω.

Definition 2.17.
The support of a measure $\mu : \mathfrak{B}(\Omega) \to [0, \infty]$ is the set

$$\operatorname{supp}\mu := \{\omega \in \Omega : \mu(N_\omega) > 0,\ \forall N_\omega \in \tau_\Omega\}.$$

Lemma 2.18.
With notation as above, the following hold.

1. $\operatorname{supp}\mu$ *is a closed (and thus Borel-measurable) subset of* Ω*.*
2. *If* Ω *is Hausdorff, and* μ *is a Radon measure, every Borel-measurable set* $S \subset \Omega \setminus \operatorname{supp}\mu$ *has* $\mu(S) = 0$*.*
3. *If* μ *is invariant under a continuous map* $T : \Omega \to \Omega$*, then* $\operatorname{supp}\mu$ *is also invariant,*
$$T^{-1}(\operatorname{supp}\mu) \subseteq \operatorname{supp}\mu.$$

Existence of invariant measures

Theorem 2.19 (Krylov-Bogoliubov).
Let (Ω, τ_Ω) *be a compact metrizable space and* $T : \Omega \to \Omega$ *a continuous map. Then, there exists an invariant Borel probability measure under* T*.*

Existence of dense orbits

Theorem 2.20.
Let (Ω, τ_Ω) *be a compact metrizable space,* $T : \Omega \to \Omega$ *a continuous map, and* μ *an ergodic, invariant Borel probability measure with* $\operatorname{supp} \mu = \Omega$*. Then,* μ*-a.e.* $\omega \in \Omega$ *has a dense orbit* $\{T^n(\omega)\}_{n=0}^{\infty}$*.*

Geometry of invariant measures

Theorem 2.21.
Let $T : \Omega \to \Omega$ be a continuous map on a compact metrizable space. Let $\mathcal{M}(\Omega; T)$ denote the set of T-invariant Borel probability measures on Ω. Then, the following hold:

1. *$\mathcal{M}(\Omega; T)$ is a weak-* compact, convex space.*
2. *μ is an extreme point of $\mathcal{M}(\Omega; T)$ iff it is ergodic.*
3. *If μ and ν are distinct, ergodic measures in $\mathcal{M}(\Omega; T)$, then they are mutually singular.*

Equidistributed sequences

Definition 2.22.
Let $T : \Omega \to \Omega$ be a continuous map on a compact metrizable space (Ω, τ_Ω) and μ a Borel probability measure. A sequence $\omega_0, \omega_1, \ldots$ with $\omega_n = T^n(\omega_0)$ is said to be μ-equidistributed if

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(\omega_n) = \int_\Omega f \, d\mu, \quad \forall f \in C(\Omega).$$

Remark.
μ-equidistribution of $\omega_0, \omega_1, \ldots$ is equivalent to weak-* convergence of the sampling measures $\mu_N = N^{-1} \sum_{n=0}^{N-1} \delta_{\omega_n}$ to the measure μ.

Basin of a measure

Definition 2.23.
With the notation of Definition 2.22 the basin of μ is the set

$$\mathcal{B}(\mu) = \{\omega_0 \in \Omega : \omega_0, \omega_1, \dots \text{ is } \mu\text{-equidistributed}\}.$$

By the pointwise ergodic theorem (Theorem 2.15), if Ω is a metrizable space and μ is an ergodic invariant measure with compact support, then μ-a.e. $\omega \in \Omega$ lies in $\mathcal{B}(\mu)$.

Observable measures

Definition 2.24.
With the notation of Definition 2.23, let ν be a reference Borel probability measure on Ω. The measure μ is said to be ν-observable if there exists a Borel set $S \in \mathfrak{B}(\Omega)$ with $\nu(S) > 0$ such that ν-a.e. $\omega \in S$ lies in $\mathcal{B}(\mu)$.

Intuitively, we think of ν as the measure from which we draw initial conditions. ν-observability of μ then means that the statistics of observables with respect to μ can be approximated from experimentally accessible initial conditions.

Koopman operators on spaces of continuous functions

Proposition 2.25.

Let $T : \Omega \to \Omega$ be a continuous map on a locally compact Hausdorff space. Then, the Koopman operator $U : f \mapsto f \circ T$ is well-defined as a linear map from $C(\Omega)$ into itself. Moreover:

1. *U is a contraction, i.e.,*

$$\|Uf\|_{C(\Omega)} \le \|f\|_{C(\Omega)}, \quad \forall f \in C(\Omega),$$

   with equality if T is invertible.

2. *U has operator norm $\|U\| = 1$.*
3. *U has the properties*

$$U(fg) = (Uf)(Ug), \quad U(f^*) = (Uf)^*, \quad \forall f, g \in C(\Omega),$$

   i.e., it is a *-homomorphism of the C^*-algebra $C(\Omega)$.

Transfer operators on Borel measures

Notation.

- $M(\Omega)$: Space of signed Borel measures on topological space (Ω, τ_Ω).

Definition 2.26.
Let $T : \Omega \to \Omega$ be a continuous map on a compact metrizable space. The transfer operator $P : C(\Omega)^* \to C(\Omega)^*$ is the transpose (dual) operator to the Koopman operator $U : C(\Omega) \to C(\Omega)$,

$$P\alpha = \alpha \circ U.$$

Unique ergodicity

Definition 2.27.
Let $T : \Omega \to \Omega$ be a continuous map on a compact metrizable space (Ω, τ_Ω). T is said to be uniquely ergodic if there is only one T-invariant Borel probability measure.

Theorem 2.28.
With notation as above, the following are equivalent.

1. *T is uniquely ergodic.*
2. *For every $f \in C(\Omega)$, $N^{-1} \sum_{n=0}^{N-1} f(T^n(\omega))$ converges to a constant, uniformly with respect to $\omega \in \Omega$.*
3. *For every $f \in C(\Omega)$, $N^{-1} \sum_{n=0}^{N-1} f(T^n(\omega))$ converges pointwise to a constant.*
4. *There exists an invariant Borel probability measure μ such that*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n(\omega)) = \int_\Omega f \, d\mu, \quad \forall \omega \in \Omega.$$

Strong and weak continuity of continuous-time (semi)flows

Theorem 2.29.
Let $\Phi^t : \Omega \to \Omega$, $t \geq 0$, be a continuous-time, continuous, semiflow on a compact metrizable space Ω with associated Koopman operators $U^t : C(\Omega) \to C(\Omega)$. Then, as $t \to 0$, U^t converges strongly to the identity,

$$\lim_{t \to 0} \| U^t f - f \|_{C(\Omega)} = 0, \quad \forall f \in C(\Omega).$$

Theorem 2.30.
Let $\Phi^t : \Omega \to \Omega$, $t \geq 0$, be a continuous-time, measurable semiflow with invariant probability measure μ and associated Koopman operators $U^t : L^p(\mu) \to L^p(\mu)$. Then, the following hold as $t \to 0$:

1. *For $1 \leq p < \infty$, U^t converges strongly to the identity,*

$$\lim_{t \to 0} \| U^t f - f \|_{L^p(\mu)} = 0, \quad \forall f \in L^p(\mu).$$

2. *For $p = \infty$, U^t converges in weak-* sense to the identity,*

$$\lim_{t \to 0} \int_\Omega g(U^t f)\, d\mu = \int_\Omega gf\, d\mu, \quad \forall f \in L^\infty(\mu), \quad \forall g \in L^1(\mu).$$

Mixing

Recall from Theorem 2.4 that a measure-preserving transformation is ergodic iff

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu(T^{-n}(R) \cap S) = \mu(R)\mu(S), \quad \forall R, S \in \Sigma.$$

Definition 2.31.
Let $T : \Omega \to \Omega$ be a measure-preserving transformation of the probability space (Ω, Σ, μ).

1. T is said to be weak-mixing if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |\mu(T^{-n}(R) \cap S) - \mu(R)\mu(S)| = 0, \quad \forall R, S \in \Sigma.$$

2. T is said to be strong-mixing, or mixing, if

$$\lim_{n\to\infty} \mu(T^{-n}(R) \cap S) = \mu(R)\mu(S), \quad \forall R, S \in \Sigma.$$

Mixing

Theorem 2.32.
Let $T : \Omega \to \Omega$ be a measure-preserving transformation of the probability space (Ω, Σ, μ). Then, the following are equivalent.

1. *T is weak-mixing.*
2. *There is a subset $\mathcal{N} \subset \mathbb{N}$ of zero density such that*

$$\lim_{\substack{n \to \infty \\ n \notin \mathcal{N}}} \mu(T^{-n}(R) \cap S) = \mu(R)\mu(S), \quad \forall R, S \in \Sigma.$$

Observable-centric characterization of ergodicity and mixing

Let $T : \Omega \to \Omega$ be a measure-preserving transformation of the probability space (Ω, Σ, μ). Let $U : L^2(\mu) \to L^2(\mu)$ be the associated Koopman operator on L^2.

For $f, g \in L^2(\mu)$, define the cross-correlation function $C_{fg} : \mathbb{N} \to \mathbb{R}$, where

$$C_{fg}(n) = \langle f, U^n g \rangle_{L^2(\mu)},$$

and the autocorrelation function $C_f = C_{ff}$.

Consider also the expectation values $\bar{f} = \int_\Omega f \, d\mu$ and $\bar{g} = \int_\Omega g \, d\mu$.

Theorem 2.33.
With notation as above, the following are equivalent.

1. *T is ergodic.*
2. *For all $f, g \in L^2(\mu)$, $\lim_{n\to\infty} N^{-1} \sum_{n=0}^{N-1} C_{fg}(n) = \bar{f}\bar{g}$.*
3. *For all $f \in L^2(\mu)$, $\lim_{n\to\infty} N^{-1} \sum_{n=0}^{N-1} C_f(n) = \bar{f}^2$.*

Observable-centric characterization of ergodicity and mixing

Theorem 2.34.
With notation as above, the following are equivalent.

1. T *is weak-mixing.*
2. *For all* $f, g \in L^2(\mu)$, $\lim_{N\to\infty} N^{-1} \sum_{n=0}^{N-1} |C_{fg}(n) - \bar{f}\bar{g}| = 0$.
3. *For all* $f \in L^2(\mu)$, $\lim_{N\to\infty} N^{-1} \sum_{n=0}^{N-1} |C_f(n) - \bar{f}^2| = 0$.

Theorem 2.35.
With notation as above, the following are equivalent.

1. T *is mixing.*
2. *For all* $f, g \in L^2(\mu)$, $\lim_{N\to\infty} C_{fg}(n) = \bar{f}\bar{g}$.
3. *For all* $f \in L^2(\mu)$, $\lim_{N\to\infty} C_f(n) = \bar{f}^2$.

Spectral characterization of mixing

Theorem 2.36.
Let $T : \Omega \to \Omega$ be a measure-preserving transformation of the probability space (Ω, Σ, μ), and $U : L^2(\mu) \to L^2(\mu)$ the corresponding Koopman operator. Then, T is weak-mixing iff the only eigenvalue of U is the eigenvalue equal to 1.

Mixing and product flows

Theorem 2.37.
Let $T : \Omega \to \Omega$ be a measure-preserving transformation of the probability space (Ω, Σ, μ). Then, the following are equivalent.

1. *T is weak-mixing.*
2. *$T \times T$ is ergodic with respect to the product measure $\mu \times \mu$.*
3. *$T \times T$ is weak-mixing with respect to the product measure $\mu \times \mu$.*

Further reading

[1] V. Baladi, *Positive Transfer Operators and Decay of Correlations* (Advanced Series in Nonlinear Dynamics). Singapore: World Scientific, 2000, vol. 16.

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