Section 3

Introduction to operator algebras

Algebras – basic definitions

Definition 3.1.
An algebra (over the complex numbers) is a $\mathbb{C}$-vector space $\mathcal{A}$, equipped with a binary operation $\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ such that for every $a, b, c \in \mathcal{A}$ and $\lambda \in \mathbb{C}$, we have:

- $(ab)c = a(bc)$.
- $a(b + \cancel{1}^{\,c}) = ab + ac$.
- $(a + b)c = ac + bc$.
- $(\lambda a)b = \lambda(ab) = a(\lambda b)$.

Examples

- $\mathbb{C}^n$ with $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} a_1 b_1 \\ \vdots \\ a_n b_n \end{pmatrix}$

  abelian, unital with $\mathbb{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

  Banach algebra wrt any vector norm

  C^*-alg. wrt max. norm

- $L^1(\mathbb{R})$ with $(f * g)(t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau)\, d\tau$

  Check that indeed $f*g$ lies in $L^1(\mathbb{R})$ whenever $f, g \in L^1(\mathbb{R})$

  Abelian but non-unital

  Banach wrt L^1 norm

- M_n ($n \times n$ complex matrices) equipped with matrix product

  Banach wrt any matrix norm, C^*-alg. wrt. matrix 2-norm

  Composition of linear maps from $\mathbb{C}^n \to \mathbb{C}^n$

  non-abelian, unital with $\mathbb{1} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

- Given a probability space (Ω, Σ, μ),

  $L^\infty(\mu)$ equipped with pointwise function multiplication

  ↳ abelian, unital

  C^*-alg. wrt ∞ norm.

- Given a Hilbert space H (e.g., $H = L^2(\mu)$)

  $B(H)$ equipped with operator composition

  ↳ non abelian unital

  C^*-algebra wrt operator norm

  $K(H) \subseteq B(H)$ compact operators

  ↳ non abelian, non-unital

- Given a Hausdorff top. space X,

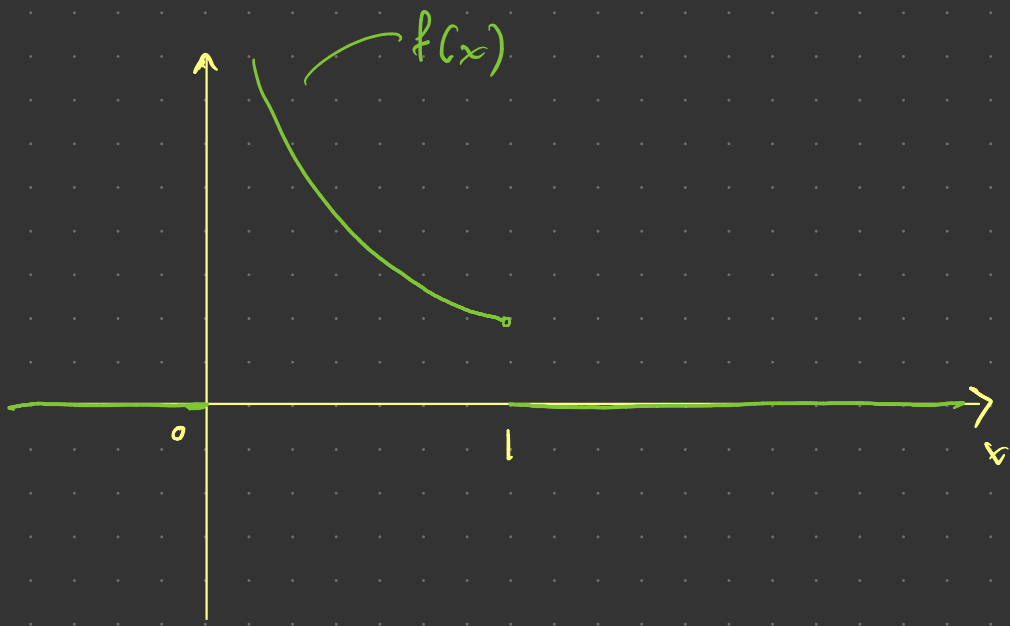
  $C_0(X)$ (continuous functions that vanish at ∞) equipped with pointwise function multiplication

  C^*-algebra equipped with ∞ norm

  ↳ abelian, non-unital when X is non-compact

Non-examples

$L^1(\mathbb{R})$ under pointwise multiplication of functions is not an algebra



$$f(x) = \begin{cases} 0, & x \leq 0 \\ 1/\sqrt{x}, & x \in (0,1) \\ 0, & x \geq 1 \end{cases}$$

Then $\int_{\mathbb{R}} f(x)\,dx < \infty$

but $\int_{\mathbb{R}} f^2(x)\,dx = \infty$

Algebras – basic definitions

Definition 3.2.
An algebra $\mathcal{A}$ is said to be:

1. Abelian if $ab = ba$ for all $a, b \in \mathcal{A}$.
2. Unital if there is a (unique) nonzero element $\mathbb{1} \in \mathcal{A}$ such that $\mathbb{1}a = a\mathbb{1} = \underset{a}{\cancel{\mathbb{1}}}$ for all $a \in \mathcal{A}$.

*-algebras

Definition 3.3.
A *-algebra (or involutive algebra) is an algebra $\mathcal{A}$ equipped with an operation $^* : \mathcal{A} \to \mathcal{A}$ such that for all $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{C}$,

- $(a^*)^* = a$.
- $(a + b)^* = a^* + b^*$.
- $(ab)^* = b^* a^*$.
- $(\lambda a)^* = \lambda^* a^*$.

<u>Examples</u> $\mathcal{A} = \mathbb{C}^n$ $\qquad \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}^* = \begin{pmatrix} a_1^* \\ \vdots \\ a_n^* \end{pmatrix}$

$\mathcal{A} = M_n$ $\qquad A^* \sim$ complex-conjugate transpose

$L^\infty(\mu) \sim f^*(x) = (f(x))^*$

$B(H)$ $\qquad A^* \sim$ operator adjoint $\left(\langle f, Ag \rangle_H = \langle A^* f, g \rangle_H \quad \forall f, g \in H \right)$

Banach algebras; C^*-algebras

Implies that multiplication is continuous as a map from $\mathcal{A}\times\mathcal{A}$ to $\mathcal{A}$.

Definition 3.4.

1. A normed algebra is an algebra $\mathcal{A}$ equipped with a norm $\|\cdot\|$ such that
$$\|ab\| \leq \|a\|\|b\|, \quad \forall a, b \in \mathcal{A}.$$
2. A Banach algebra is a normed algebra $(\mathcal{A}, \|\cdot\|)$ which is complete with respect to $\|\cdot\|$.
3. A C^*-algebra is a Banach *-algebra such that
$$\underbrace{\|a^* a\| = \|a\|^2}_{\text{"}C^*\text{-identity"}}.$$

For a unital normed algebra, we can choose the norm such that $\|\mathbb{1}\| = 1$ without loss of generality.

Definition 3.5.

1. Given an algebra $\mathcal{A}$, then for a subset $S \subseteq \mathcal{A}$ we denote by $\operatorname{alg}(S)$ the subalgebra of $\mathcal{A}$ generated by S, which consists of all linear combinations of finite products of elements of S. Equivalently, $\operatorname{alg}(S)$ is the smallest subalgebra of $\mathcal{A}$ containing S.
2. If $\mathcal{A}$ is a Banach algebra, the closure $\overline{\operatorname{alg}(S)}$ is said to be the Banach subalgebra of $\mathcal{A}$ generated by S.

Inverse

Definition 3.6.
An element a of a unital algebra $\mathcal{A}$ is said to be invertible if there exists a (unique) element $b \in \mathcal{A}$ such that $ab = ba = \mathbb{1}$. We write $b = a^{-1}$ and call a^{-1} the inverse of a.

We denote the set of invertible elements of $\mathcal{A}$ as $G(\mathcal{A})$. This set forms a multiplicative subgroup of $\mathcal{A}$.

Proposition 3.7.
For a unital Banach algebra $\mathcal{A}$, $G(\mathcal{A})$ is an open set and $^{-1} : G(\mathcal{A}) \to \mathcal{A}$ is continuous. ~~*Henceforth, we shall assume that this is the case.*~~

Proposition Let A be a unital Banach algebra and let $a \in A$ have norm $\|a\| < 1$. Then $b = \mathbb{1} - a$ is invertible.

Sketch of proof. Postulate that the inverse of b is equal to

$$c = \mathbb{1} + a + a^2 + \dots$$

Check that (i) the series $c_N = \sum_{n=0}^{N-1} a^n$ converges whenever $\|a\| < 1$

$$\left(\text{compute } \|c - c_N\| = \left\| \sum_{n=N}^{\infty} a^n \right\| \leq \sum_{n=N}^{\infty} \|a\|^n \right)$$

(ii) Check that $cb = bc = \mathbb{1}$.

Normal elements

Definition 3.8.
An element a of a *-algebra is said to be:

1. Normal if it commutes with a^*, i.e., $aa^* - a^*a = 0$.
2. Self-adjoint if $a^* = a$. e.g. self-adjoint elements of $L^\infty(\mu)$ are real-valued functions
3. Skew-adjoint if $a^* = -a$.

Given a unital Banach algebra $\mathcal{A}$ and an element $a \in \mathcal{A}$ we denote the Banach algebra generated by $\{\mathbb{1}, a\}$ as $B(a)$. If, in addition, $\mathcal{A}$ is a *-algebra, we let $B^*(a)$ be the Banach *-algebra generated by $\{\mathbb{1}, a, a^*\}$.

abelian

Lemma 3.9.
If $a \in \mathcal{A}$ is a normal element of a Banach *-algebra, then $B^*(a)$ is abelian.

Spectrum

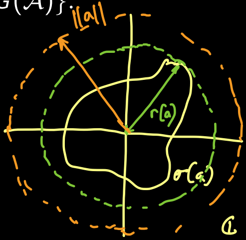
Definition 3.10.
For an element $a \in \mathcal{A}$ of a unital Banach algebra $\mathcal{A}$ we define:

1. The spectrum as the set of complex numbers

$$\sigma(a) = \{\lambda \in \mathbb{C} : a - \underbrace{\lambda}_{\equiv \lambda \mathbb{1}} \notin G(\mathcal{A})\}.$$

2. The spectral radius

$$r(a) = \sup_{\lambda \in \sigma(a)} |\lambda|.$$



Theorem 3.11.
With notation as above, the following hold:

1. $\sigma(a)$ *is a compact subset of* $\mathbb{C}$ *such that*

$$\sup_{\lambda \in \sigma(a)} |\lambda| \leq \|a\|.$$

2. $r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$.
3. *If a is a normal element of a C^*-algebra, then $r(a) = \|a\|$.*

Ex. $\mathcal{A} = M_2$

$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\sigma(a) = \{0\}$
$\Rightarrow r(a) = 0$

$\|a\| = 1$

Examples of spectra

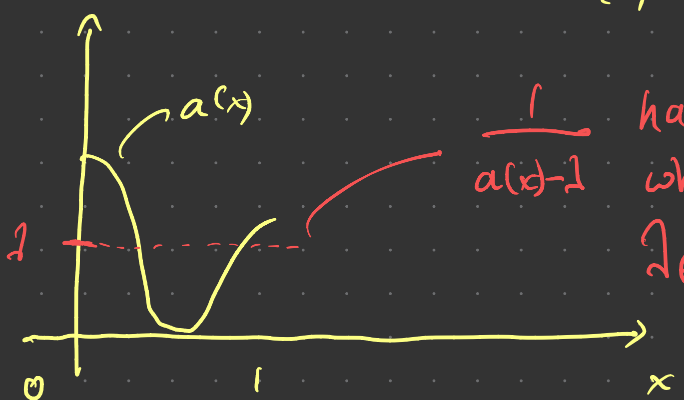
1) $\mathcal{A} = \mathbb{M}_n$ $\quad \sigma(a) = \{\text{set of eigenvalues of } a\}$

2) $\mathcal{A} = C(X)$, X compact Hausdorff

$\sigma(a) = \{\lambda \in \mathbb{C} :$ $a - \lambda$ is not invertible as a continuous function, i.e.,

$b(x) = \dfrac{1}{a(x) - \lambda}$ is not a continuous function $\}$

$= \operatorname{ran} a$



$\dfrac{1}{a(x)-\lambda}$ has singularities whenever $\lambda \in \operatorname{ran} a$

Homomorphisms

Definition 3.12.

1. A homomorphism $\pi : \mathcal{A} \to \mathcal{B}$ between algebras is a linear map that is compatible with algebraic multiplication, i.e.,

$$\pi(aa') = \pi(a)\pi(a'), \quad \forall a, a' \in \mathcal{A}.$$

2. A homomorphism $\pi : \mathcal{A} \to \mathcal{B}$ is said to be unital if $\mathcal{A}$ and $\mathcal{B}$ are unital and $\pi(\mathbb{1}_{\mathcal{A}}) = \mathbb{1}_{\mathcal{B}}$.
3. A homomorphism $\pi : \mathcal{A} \to \mathcal{B}$ between *-algebras is said to be a *-homomorphism if

$$\pi(a^*) = (\pi a)^*, \quad \forall a \in \mathcal{A}.$$

Examples

$\mathcal{A} = \mathbb{C}^n$, $\mathcal{B} = M_n$ $\quad \pi \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}$

$$\pi\left(\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \right) = \begin{pmatrix} a_1 b_1 & & 0 \\ & \ddots & \\ 0 & & a_n b_n \end{pmatrix} = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \begin{pmatrix} b_1 & & 0 \\ & \ddots & \\ 0 & & b_n \end{pmatrix} = \pi \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \pi \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$\mathcal{A} = L^\infty(\mu)$ $\quad \mathcal{B} = B(L^2(\mu))$

$\pi f = A$ s.t. $Ag = fg$.

Can show: $\|A\|_{\mathcal{B}} = \|f\|_{\mathcal{A}}$

Representations

Definition 3.13.

1. For an algebra $\mathcal{A}$, a representation is a homomorphism $\pi : \mathcal{A} \to L(V)$, where $L(V)$ is the algebra of linear maps on a vector space V.
2. If $\mathcal{A}$ is a Banach algebra, a representation is a homomorphism $\pi : \mathcal{A} \to B(E)$, where $B(E)$ is the Banach algebra of bounded linear maps on a Banach space E.
3. If $\mathcal{A}$ is a Banach *-algebra, a *-representation is a $*$-homomorphism $\pi : \mathcal{A} \to B(H)$, where $B(H)$ is the C^*-algebra of bounded linear maps on a Hilbert space H.
4. If $\ker \pi = \{0\}$, π is said to be a faithful representation.

Representations

Definition 3.14.
For a Banach algebra $\mathcal{A}$, the left regular representation (or left multiplier representation) $\pi : A \to B(\mathcal{A})$ is defined as

$$(\pi a)b = ab, \quad \forall a, b \in \mathcal{A}.$$

Proposition 3.15.

1. *The left regular representation $\pi : A \to L(\mathcal{A})$ of a unital algebra $\mathcal{A}$ is faithful.*
2. *If $\mathcal{A}$ is a Banach algebra, then π is a contraction; that is, $\|\pi\| \leq 1$.*
3. *If $\mathcal{A}$ is a C^*-algebra, then π is an isometry; that is, $\|\pi\| = 1$.*

Representations of C^*-algebras

Lemma 3.16.
Let H be a Hilbert space. Then, any norm-closed *-subalgebra $\mathcal{A}$ of $B(H)$ is a C^*-algebra. We refer to every such $\mathcal{A}$ as a concrete C^*-algebra.

Theorem 3.17 (Gelfand–Naimark–Segal).
Every C^*-algebra $\mathcal{A}$ admits admits a faithful representation $\pi : \mathcal{A} \to B(H)$ on some Hilbert space H.

$\pi \mathcal{A}$ is a concrete C^*-algebra.

$H = \mathbb{C}^n$, equipped with Euclidean inner prod.

$B(\mathbb{C}^n) \simeq M_n$. $\mathcal{A} = \mathbb{D}_n$ is a C^*-subalgebra of $B(\mathbb{C}^n)$

$n \times n$ diag. matrices

$H = L^2(\mu)$ $\mathcal{A} = \{$ mult. operators by $L^\infty(\mu)$ funcs. $\}$ is a C^*-subalgebra of $B(L^2(\mu))$

$K(H)$ (compact operators on H) is a C^*-subalgebra of $B(H)$, H Hilbert space.

Characters

Definition 3.18.
A character (or multiplicative linear functional) of a unital Banach algebra $\mathcal{A}$ is a nonzero homomorphism $\chi : \mathcal{A} \to \mathbb{C}$.

Lemma 3.19.
Every character $\chi : \mathcal{A} \to \mathbb{C}$ is:

1. *Unital.*
2. *Surjective.*
3. *Contractive, i.e., $\|\chi\| \leq 1$.*

Moreover, if $\mathcal{A}$ is a C^*-algebra, then:

4. *χ is a *-homomorphism.*
5. *$\|\chi\| = 1$.*

Corollary 3.20.
Every character of a unital Banach algebra is continuous.

↳ characters lie in the continuous dual $\mathcal{A}^*$

Example: $\mathcal{A} = C(X)$, X compact, Hausdorff

For any $x \in X$ let $\delta_x : \mathcal{A} \to \mathbb{C}$ be the evaluation functional at x, i.e.,

$$\delta_x f = f(x).$$

Then, δ_x is a character: $\delta_x(fg) = f(x)g(x) = (\delta_x f)(\delta_x g)$

In fact, every character of $C(X)$ is of this form.

Non-example: $\mathcal{A} = M_n$ has no characters.

Indeed, let $e_{ij} \in M_n$ be the matrix whose only nonzero element is $(e_{ij})_{ij} = 1$.

Then whenever $i \neq j$, $e_{ij}^2 = 0$. Thus, any character χ would satisfy

$0 = \chi(e_{ij}^2) = (\chi(e_{ij}))^2 \Rightarrow \chi(e_{ij}) = 0$.

However, we also have $e_{ij}e_{ji} = e_{ii}$. Thus, $\mathbb{1} = e_{11} + \dots + e_{nn}$

$$= e_{1j_1}e_{j_11} + \dots + e_{nj_n}e_{j_nn}$$

where $j_i \neq i$, and we would have $\chi(\mathbb{1}) = 0$ which is not possible

Characters

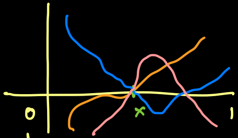
Proposition 3.21.
An abelian unital Banach algebra has at least one character.

Ideals

Definition 3.22.
A subalgebra $\mathcal{I} \subseteq \mathcal{A}$ of an algebra is said to be a (two-sided) ideal if $a\mathcal{I} \subseteq \mathcal{I}$ and $\mathcal{I}a \subseteq \mathcal{I}$ for all $a \in \mathcal{A}$.

Definition 3.23.
A maximal ideal is a proper ideal $\mathcal{I} \subset \mathcal{A}$ that is not a subset of any other proper ideals.

Proposition 3.24.
Every maximal ideal in a unital Banach algebra is closed.



Examples (i) $\mathcal{A} = C(X)$

For an $x \in X$,

$\mathcal{I}_x = \{f \in C(X) : f(x) = 0\} = f \in \ker \delta_x$

is a maximal ideal

(ii) $\mathcal{A} = B(H)$. Then $K(H)$ is an ideal in $B(H)$

Spectra of abelian Banach algebras

Definition 3.25.
Let $\mathcal{A}$ be a unital, abelian Banach algebra. The spectrum of $\mathcal{A}$, denoted as $\sigma(\mathcal{A})$, is the set of its characters.

Theorem 3.26 (Gelfand–Mazur).
Let $\mathcal{A}$ be an abelian unital Banach algebra. There is a canonical bijection between $\sigma(\mathcal{A})$ and the set of maximal ideals of $\mathcal{A}$. Specifically, for every $\chi \in \sigma(\mathcal{A})$, $\ker \chi$ is a maximal ideal, and every maximal ideal has this form for a unique character $\chi \in \sigma(\mathcal{A})$.

Gelfand transform

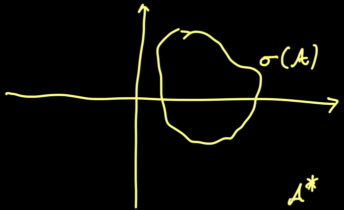
C*-algebra by weak-* compactness of $\sigma(\mathcal{A})$

Theorem 3.27.
The spectrum $\sigma(\mathcal{A})$ of an abelian unital Banach algebra is a weak- compact subset of $\mathcal{A}^*$. Moreover, the map $\hat{}: \mathcal{A} \to C(\sigma(\mathcal{A}))$ with $\hat{a}(\chi) = \chi(a)$ is a Banach algebra homomorphism with norm $\|\hat{}\| \leq 1$.*

Definition 3.28.
The map $\hat{}: \mathcal{A} \to C(\sigma(\mathcal{A}))$ is called the Gelfand transform for $\mathcal{A}$.

(sometimes we write $\hat{a} \equiv \Gamma(a)$)

Proposition 3.29.
The Gelfand transform for $\mathcal{A}$ is injective iff the intersection of all the maximal ideals of $\mathcal{A}$ is $\{0\}$. In that case, we say that $\mathcal{A}$ is semisimple.



Recall: a sequence $\alpha_1, \alpha_2, \dots \in \mathcal{A}^*$ converges in the weak-* topology to $\alpha \in \mathcal{A}^*$ if for every $f \in \mathcal{A}$, $\lim_{n\to\infty} \alpha_n f = \alpha f$.

$\sigma(\mathcal{A})$ is weak *-compact $\Leftrightarrow$ every open cover of $\sigma(\mathcal{A})$ in weak-* topology has a finite subcover.

Gelfand transform

Proposition 3.30.
For an element a of an abelian, unital, Banach algebra $\mathcal{A}$ we have

$$\sigma(a) = \operatorname{ran} \hat{a} = \sigma(\cancel{B}(a)).$$

$\{\lambda \in \mathbb{C} : a - \lambda \notin G(\mathcal{A})\}$

$\{\lambda \in \mathbb{C} : \chi(a) = \lambda$ for some $\chi \in \sigma(\mathcal{A})\}$

Prop. Let $\mathcal{A}$ be a unital, abelian Banach algebra generated by $\{\mathbb{1}, a\}$. Then, $\beta : \sigma(\mathcal{A}) \to \sigma(a)$ defined as $\beta(\chi) = \hat{a}(\chi)$ is a homeomorphism between the spectrum of $\mathcal{A}$ and the spectrum a.

Spectra of C^*-algebras

Theorem 3.32 (Gelfand).

Let $\mathcal{A}$ be a unital, abelian C^*-algebra. Then, the Gelfand transform $\Gamma : \mathcal{A} \to C(\sigma(\mathcal{A}))$ is an isometric *-isomorphism between $\mathcal{A}$ and the C^*-algebra of continuous functions on $\sigma(\mathcal{A})$.

Theorem 3.33 (Stone).

Let X be a compact Hausdorff space. For $x \in X$ let $\delta_x \in C(X)^*$ denote the evaluation functional $\delta_x f = f(x)$. Then, the following hold.

1. $\sigma(C(X)) = \{\delta_x : x \in X\}$.
2. *X is homeomorphic to $\sigma(C(X))$ under the map $x \mapsto \delta_x$.*

Corollary 3.34.

Let X and Y be compact Hausdorff spaces. Then, X and Y are homeomorphic iff $C(X)$ and $C(Y)$ are algebraically isomorphic. In that case, $C(X)$ and $C(Y)$ are isometrically *-isomorphic C^*-algebras.

Spectra of C^*-algebras

Based on Theorems 3.32 and 3.33, we can identify unital abelian C^*-algebras with spaces of continuous functions on compact Hausdorff spaces. Generalizing this interpretation, we can interpret non-abelian C^* algebras as spaces of continuous functions on "non-commutative spaces".

Continuous functional calculus

Let a be a normal element of a unital C^*-algebra $\mathcal{A}$. Given a continuous function $f : \sigma(a) \to \sigma(a)$, we define $f(a) \in \mathcal{A}$ as

$$f(a) = \Gamma^{-1}(f \circ \beta),$$

where $\Gamma : C^*(a) \to C(\sigma(C^*(a)))$ is the Gelfand transform associated with the abelian C^*-algebra generated by a, and $\beta : \sigma(C^*(a)) \to \sigma(a)$ is the homeomorphism from Proposition 3.31.

Positive elements

Definition 3.35.
An element a of a *-algebra $\mathcal{A}$ is said to be positive if $a = b^* b$ for some $b \in \mathcal{A}$.

Definition 3.36.
A *-algebra $\mathcal{A}$ is said to be:

1. Hermitian if every self-adjoint element has real spectrum, i.e., $a \in \mathcal{A}$ and $a^* = a$ implies $\sigma(a) \subset \mathbb{R}$.
2. Symmetric if every positive element has positive spectrum, i.e., $a \in \mathcal{A}$ and $a \geq 0$ implies $\sigma(a) \subset \mathbb{R}_+$.

Theorem 3.37.
A Banach *-algebra $\mathcal{A}$ is Hermitian iff it is symmetric.

Examples

(1) $\mathcal{A} = M_n(\mathbb{C})$. Linear algebra result:

The following are equivalent:

(i) $a = b^*b$ for $a, b \in M_n(\mathbb{C})$

(ii) $\forall \xi \in \mathbb{C}^n$, $\langle \xi, a\xi \rangle \equiv \xi^\dagger a \xi \geq 0$

↳ complex conj. transpose

For every such matrix a, $\sigma(a) \subset \mathbb{R}_+$.

⚠ For a general matrix a $\sigma(a) \subset \mathbb{R}_+$ does not imply that a is positive, e.g.,

$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ has $\sigma(a) = \{0\}$ but it is not positive.

(2) $\mathcal{A} = C(X)$, X compact Hausdorff

The following are equivalent:

(i) $f = g^*g$ for $f, g \in C(X)$

(ii) $\forall x \in X$ $f(x) \geq 0$.

Positive elements of C^*-algebras

Theorem 3.38.
Let $\mathcal{A}$ be a C^*-algebra. The following are equivalent:

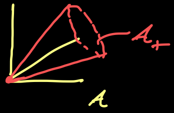
1. *a is positive (i.e., $a = b^*b$ for some $b \in \mathcal{A}$).*
2. *a is normal and $\sigma(a) \subset [0, \infty)$.*
3. *There exists a self-adjoint element $b \in \mathcal{A}$ such that $a = b^2$.*

Corollary 3.39.
Every positive element $a \in \mathcal{A}$ has a unique positive square root, i.e., a positive element $b \in \mathcal{A}$ such that $a = b^2$. We write $b = \sqrt{a}$.

Notation.
For a C^*-algebra $\mathcal{A}$:

- $\mathcal{A}_{\mathrm{sa}} \subset \mathcal{A}$ is the subspace of the self-adjoint adjoint elements.
- $\mathcal{A}_+ \subset \mathcal{A}_{\mathrm{sa}}$ is the subset of positive elements.

Positive elements of C^*-algebras



Theorem 3.40.
The set of positive elements of a C^* algebra is a convex cone, i.e.,

1. *For all $a \in \mathcal{A}_+$ and $\lambda \geq 0$, $\lambda a \in \mathcal{A}_+$.*
2. *For all $a, b \in \mathcal{A}_+$ and $\lambda \in [0, 1]$, $\lambda a + (1 - \lambda b) \in \mathcal{A}_+$.*

Moreover, $\mathcal{A}_+$ is closed in the norm topology of $\mathcal{A}$.

By Theorem 3.40, positivity defines an order on $\mathcal{A}_{\mathrm{sa}}$.

- If $a \in \mathcal{A}_{\mathrm{sa}}$ is positive, we write $a \geq 0$.
- Given $a, b \in \mathcal{A}_{\mathrm{sa}}$, we write $a \leq b$ if $b - a \geq 0$.

Proposition 3.41.
Given two positive elements $a, b \in \mathcal{A}_+$ with $a \leq b$ the following hold:

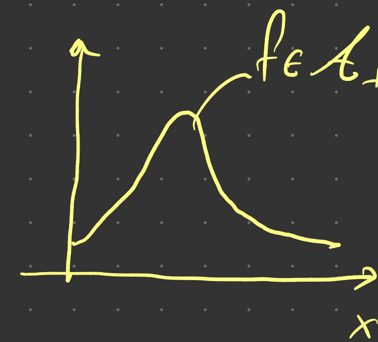
1. $\|a\| \leq \|b\|$.
2. $\sqrt{a} \leq \sqrt{b}$.
3. *If $\mathcal{A}$ is unital and a, b are invertible, then $b^{-1} \leq a^{-1}$.*

Example Approximation of multiplication operators

(X, Σ, μ) – probability measure space

$\mathcal{A} = L^\infty(\mu)$ (abelian C^*-algebra under pointwise function multiplication)

"space of classical observables"



$\mathcal{A}_{sa} = \{ f \in L^\infty(\mu) : f(x) \text{ is real for } \mu\text{-a.e. } x \in X \}$

$\mathcal{A}_+ = \{ f \in \mathcal{A}_{sa} : f(x) \geq 0 \text{ for } \mu\text{-a.e. } x \in X \}$

$H = L^2(\mu)$, $\mathcal{B} = B(H)$ (non-abelian C^*-algebra under operator composition)

"space of quantum observables"

$\mathcal{B}_{sa} = \{ a \in B(H) : \langle \eta, a\xi \rangle_H = \langle a\eta, \xi \rangle, \forall \eta, \xi \in H \}$

$\mathcal{B}_+ = \{ a \in \mathcal{B}_{sa} : \langle \xi, a\xi \rangle_H \geq 0, \forall \xi \in H \}$

Important property: $\sigma(f) = \sigma(\pi f)$, $\forall f \in \mathcal{A}$.

$\pi : \mathcal{A} \to \mathcal{B}$ regular rep. $\pi f = a$ where $a\xi = f\xi$ $\forall \xi \in H$

recall $\pi(f^*) = (\pi f)^*$ $\pi(fg) = (\pi f)(\pi g)$.

This implies that π maps $\mathcal{A}_+$ into $\mathcal{B}_+$; since $\pi(g^* g) = \pi(g^*)\pi(g) = (\pi g)^* (\pi g) \geq 0$

Let $\Pi : H \to H$ be a projection i.e. $\Pi = \Pi^* = \Pi^2$, s.t. $\Pi \mathcal{A} \subseteq \mathcal{A}$

Then, given $f \in \mathcal{A}_+$, Πf is, in general, not positive.

However, $\Pi (\pi f) \Pi \in \mathcal{B}$ <u>is</u> positive

Check: $\forall \xi \in H$, $\langle \xi, \Pi(\pi f)\Pi \xi \rangle = \langle \Pi \xi, (\pi f) \Pi \xi \rangle \geq 0$ since $(\pi f) \geq 0$

Application: Let $\{\phi_0, \phi_1, \dots\}$ be an O-N basis of H, $\Pi_L = \mathrm{proj}_{H_L}$, $H_L = \{\phi_0, \dots, \phi_{L-1}\}$

Then $A_L := \Pi_L (\pi f) \Pi_L$ is represented by an $L \times L$ matrix with positive eigenvalues

It can be shown that for every $\lambda \in \sigma(f)$, there is a sequence of eigenvalues $\lambda_1, \lambda_2, \dots$ of $A_1, A_2, \dots$, respectively, s.t. $\lim_{L \to \infty} \lambda_L = \lambda$.

States

Definition 3.42.
A linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ on a *-algebra $\mathcal{A}$ is said to be positive if $\varphi a \geq 0$ whenever a is positive.

Definition 3.43.
A state $\varphi : \mathcal{A} \to \mathbb{C}$ on a unital *-algebra $\mathcal{A}$ is a positive, linear unital functional, i.e.:

- $\varphi(a^* a) \geq 0$ for all $a \in \mathcal{A}$.
- $\varphi \mathbb{1} = 1$.

The state space of $\mathcal{A}$ is the set of its states, denoted as $S(\mathcal{A})$.

Examples

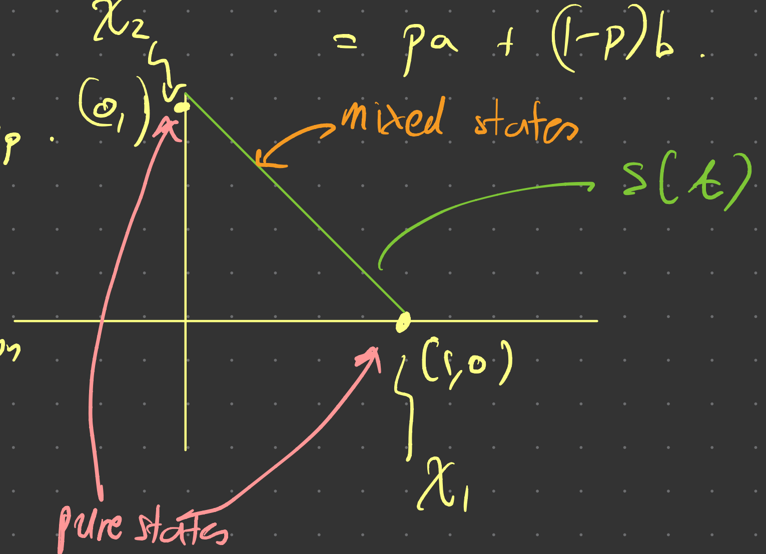
(i) $\mathcal{A}$ is an abelian unital C^*-algebra. Every character of $\mathcal{A}$ defines a state.

(ii) $\mathcal{A} = \mathbb{C}^2$. Let $\chi_1 : \mathcal{A} \to \mathbb{C}^*$ $\quad \chi_1\begin{pmatrix} a \\ b \end{pmatrix} = a, \quad \chi_2\begin{pmatrix} a \\ b \end{pmatrix} = b$

Let $p \in [0,1]$, define $\varphi_p : \mathcal{A} \to \mathbb{C}^*$ s.t. $\varphi_p\begin{pmatrix} a \\ b \end{pmatrix} = p\,\chi_1\begin{pmatrix} a \\ b \end{pmatrix} + (1-p)\,\chi_2\begin{pmatrix} a \\ b \end{pmatrix}$

$$= pa + (1-p)b.$$

Then any state of $\mathcal{A}$ is of the form φ_p.



$\dim S(\mathcal{A}) = 1$

Every state $\varphi \in S(\mathcal{A})$ has a unique decomposition $\varphi = p\chi_1 + (1-p)\chi_2$ into pure states.

↳ Every functional $\varphi : \mathcal{A} \to \mathbb{C}$ is of the form $\varphi a = q^\dagger a$ for a vector $q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \in \mathbb{C}^2$

If φ is a state, we have $q = \begin{pmatrix} p \\ 1-p \end{pmatrix}$ with $p \in [0,1]$.

(iii) $\mathcal{A} = M_2(\mathbb{C})$. $\mathcal{A}$ has the structure of a Hilbert space equipped with the inner product $\langle a, b \rangle = tr(a^* b)$. Thus, by the Riesz representation theorem, every state $\varphi : \mathcal{A} \to \mathbb{C}$ is of the form $\varphi(a) = tr(\rho a)$ for some $\rho \in \mathcal{A}$. Such a ρ is called a <u>density matrix</u>. Can verify that in order for φ to be a state we must have:

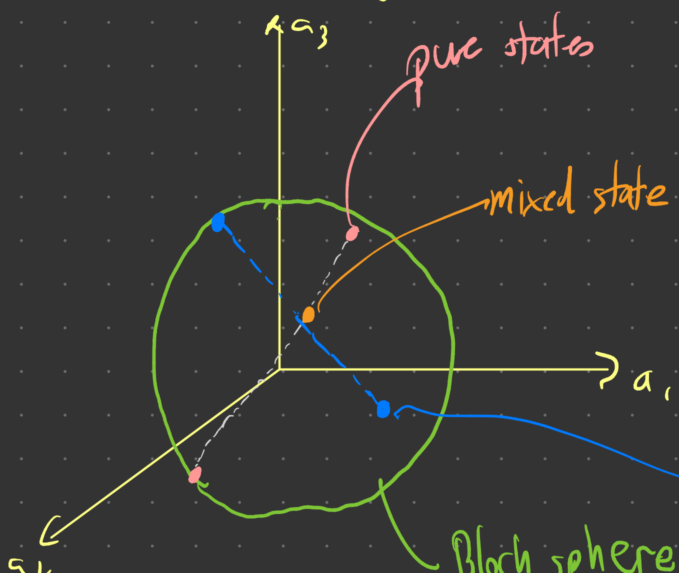
1) $\rho \geq 0$, 2) $tr\rho = 1$.

We can parameterize the set $S(\mathcal{A})$ using Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

Claim: Every 2×2 density matrix ρ can be written in the form

$$\rho = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} + a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3 \quad \text{s.t.} \quad \vec{a} = (a_1, a_2, a_3) \; \|\vec{a}\| \leq 1.$$



pure states correspond to points on surface of sphere i.e. $\|\vec{a}\| = 1$. Such density matrices are rank-1 projection matrices, i.e., $\rho = qq^T$ for a unit vector q.

mixed states are in the interior

$\dim S(\mathcal{A}) = 3$

a mixed state can be represented as linear combinations of distinct pure states.

Embedding states of $\mathbb{C}^2$ into states of M_2

(1) Map $\varphi_p \in S(\mathbb{C}^2)$ with $p \in [0,1]$ into $\varphi_\rho \in S(M_2)$ with $\rho = \begin{pmatrix} p & 0 \\ 0 & 1-p \end{pmatrix}$

$\left(T_1 : S(\mathbb{C}^2) \to S(M_2) \right)$

i.e., $\rho = \frac{1}{2} I + \frac{1}{2} p \sigma_1$

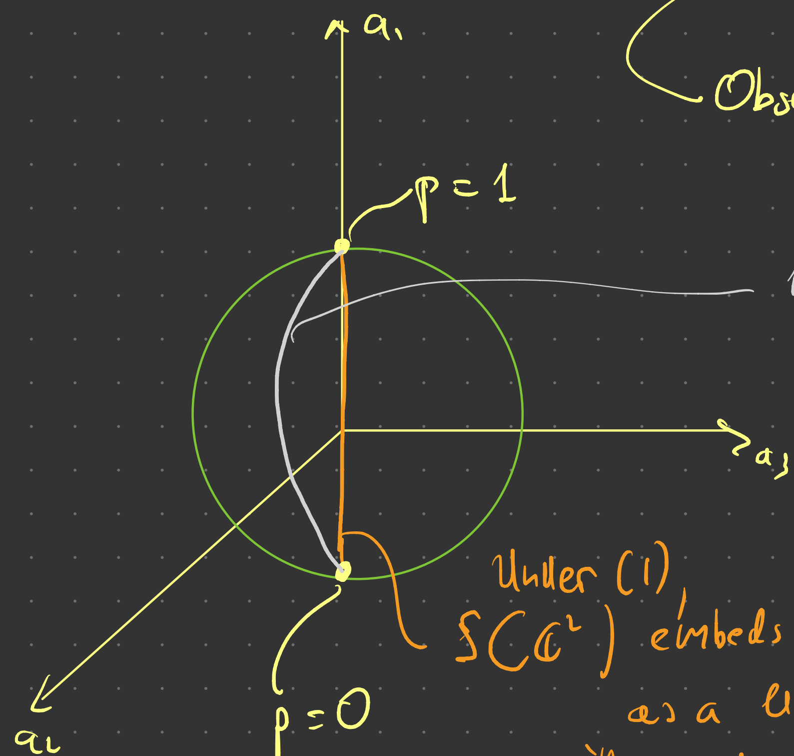
$\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(2) Map $\varphi_p \in S(\mathbb{C}^2)$ with $p \in [0,1]$ into $\varphi_\rho \in S(M_2)$ with $\rho = q q^T$

$\left(T_2 : S(\mathbb{C}^2) \to S(M_2) \right)$

$q = \begin{pmatrix} \sqrt{p} \\ \sqrt{1-p} \end{pmatrix}$.

Observe $\|q\|_2^2 = p + 1 - p = 1$.



under (2) $S(\mathbb{C}^2)$ embeds nonlinearly into $S(M_2)$ as a "meridian" connecting $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$.

Under (1), $S(\mathbb{C}^2)$ embeds linearly into $S(M_2)$ as a line through the origin with end points $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$

Embedding elements of $\mathbb{C}^2$ into elements of M_2

- We use the regular representation of $\mathbb{C}^2$, $\pi : \mathbb{C}^2 \to B(\mathbb{C}^2) \simeq M_2$

$$\pi \underbrace{\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}}_{a} = \underbrace{\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}}_{m_a}.$$

$m_a \rightsquigarrow$ multiplication operator in the sense that

$$m_a \underbrace{\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}}_{b} = \begin{pmatrix} a_1 b_1 \\ a_2 b_2 \end{pmatrix} = ab$$

"Classical quantum consistency"

For every $\varphi_p \in S(\mathbb{C}^2)$ and $a \in \mathbb{C}^2$

$$\underbrace{\varphi_p a}_{\equiv\ pa_1 + (1-p)a_2} = (\Gamma_j \varphi_p)(\pi a)$$

e.g. $$(\Gamma_1 \varphi_p)(\pi a) = tr\left(\begin{pmatrix} p & 0 \\ 0 & 1-p \end{pmatrix} \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix} \right) = tr \begin{pmatrix} pa_1 & 0 \\ 0 & (1-p)a_2 \end{pmatrix} = pa_1 + (1-p)a_2$$

To check for Γ_2, observe that for a pure state of M_2 represented by $\rho = qq^T$ we have $\varphi_\rho\, m = tr(\rho m) = tr((qq^T)m) = q^T m q \equiv \langle q | m | q \rangle$ in Dirac braket notation.

Thus, $$(\Gamma_2 \varphi_p)\pi a = q^T (\pi a)\, q = (\sqrt{p}\ \ \sqrt{1-p}) \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \begin{pmatrix} \sqrt{p} \\ \sqrt{1-p} \end{pmatrix} = pa_1 + (1-p)a_2$$

States on C^*-algebras

Proposition 3.44.

The following hold for every state $\varphi \in S(\mathcal{A})$ of a unital C^*-algebra and elements $a, b \in \mathcal{A}$.

1. $\varphi(a^*) = (\varphi a)^*$, *for all* $a \in \mathcal{A}$.
2. $|\varphi(a^* b)| \leq \varphi(a^* a)\varphi(b^* b)$.
3. $\|\varphi\| = 1$.

Proposition 3.45.

The state space $S(\mathcal{A})$ of a unital C^*-algebra $\mathcal{A}$ is a convex subset of the unit ball of $\mathcal{A}^*$ which is closed in the weak-* topology. In particular, $S(\mathcal{A})$ is a weak-* compact subset of $\mathcal{A}^*$.

$\hookrightarrow$ If $\varphi_1, \varphi_2, \ldots$ is a sequence of states such that $\forall a \in \mathcal{A}$, $\lim\limits_{n \to \infty} \varphi_n a = \varphi a$ for some $\varphi \in \mathcal{A}^*$, then φ is also a state.

States on C^*-algebras

Proposition 3.46.
positive
For every ~~self-adjoint~~ element a of a C^*-algebra $\mathcal{A}$, there exists a state $\varphi \in S(\mathcal{A})$ such that $\varphi a = \|a\|$.

Theorem 3.47.
The set of states of a unital C^*-algebra $\mathcal{A}$ separates the points of $\mathcal{A}$. That is, for every $a, b \in \mathcal{A}$ there exists $\varphi \in S(\mathcal{A})$ such that $\varphi a \neq \varphi b$.

Pure states

$\varphi \in S(\mathcal{A})$ is an extremal point if there are no $\varphi_1, \varphi_2 \in S(\mathcal{A})$ and $p \in (0,1)$ s.t. $\varphi = p\varphi_1 + (1-p)\varphi_2$.

Definition 3.48.
A state φ of a unital C^*-algebra $\mathcal{A}$ is said to be pure if it is an extremal point of $S(\mathcal{A})$. Otherwise, φ is said to be mixed.

Definition 3.49.
Let H be a Hilbert space. A state φ of $B(H)$ is said to be a vector state if there exists a (unit) vector $\xi \in H$ such that
"wavefunction"

$$\varphi a = \langle \xi, a\xi \rangle, \quad \forall a \in \mathcal{A}.$$

Proposition 3.50.
Every vector state of $B(H)$ is pure.

(since the map $a \longmapsto \langle \xi, a\xi \rangle$ is a projection, and projections are extremal points of the positive cone of $B(H)$.

If H is infinite-dimensional, there exist pure states that are not vector states.

Construction of a pure state which is not a vector state:

Let $H = L^2(X, \mu)$. Consider $\delta_x \in C(X)^*$ (evaluation functional at x).

Let $\mathcal{C} \subset B(H)$, $\mathcal{C} = \underset{\text{regular rep}}{\pi}(C(X))$, $\mathcal{C}$ consists of multiplication operators by continuous functions.

δ_x lifts to a bounded linear functional Δ_x on $\mathcal{C}$, i.e., $\Delta_x(\pi f) = \delta_x f$.

By Hahn-Banach theorem Δ_x has an extension to a continuous functional φ on $B(H)$ s.t. $\|\varphi\| = \|\Delta_x\|$ (and $\varphi(\pi f) = \delta_x f$). Such an extension φ can be chosen to be a pure state. However, for such states there is no vector $\xi \in H$ s.t. $\varphi(a) = \langle \xi, a\xi \rangle$.

Projections

Definition 3.51.
An element a of a *-algebra $\mathcal{A}$ is said to be a projection if $a = a^* = a^2$.

Proposition 3.52.
For a C^*-algebra $\mathcal{A}$, the projections are the extremal points of the positive cone $\mathcal{A}_+$.

Projection-valued measures

Definition 3.53.
Let (X, Σ) be a measurable space and H a Hilbert space. A map $E : \Sigma \to B(H)$ is said to be a projection-valued measure (PVM) if the following hold:

1. For every $S \in \Sigma$, $E(S)$ is a projection.
2. $E(\emptyset) = 0$.
3. $E(X) = I$.
4. For every countable collection $\{S_0, S_1, \ldots\}$ of pairwise-disjoint sets $S_j \in E$ and $f \in H$, we have $E(\bigcup_{j=0}^{\infty} S_j)f = \sum_{j=0}^{\infty} E(S_j)f$.

$\Updownarrow$

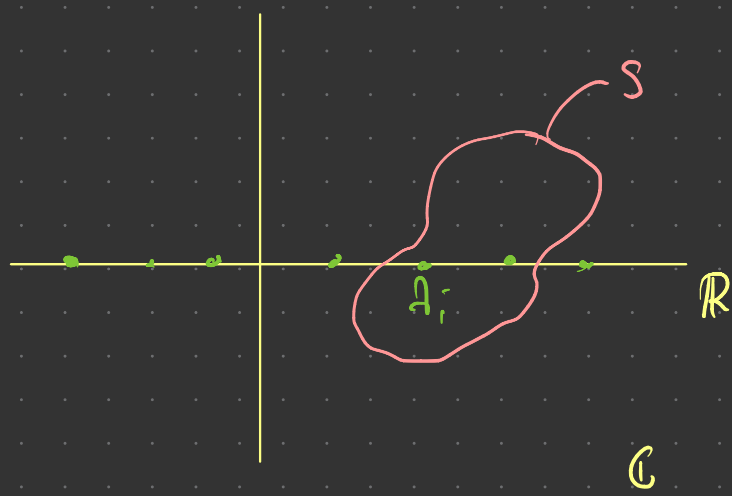
$a_J = \sum_{j=0}^{J} E(S_j)$ converges in the strong operator topology of $B(H)$.

Example $H = \mathbb{C}^n$, $B(H) \simeq M_n(\mathbb{C})$. Let $a \in M_n$ be self-adjoint.

By the spectral thm. for self-adjoint matrices, a has a set of real eigenvalues $\lambda_1, \dots, \lambda_m$ $m \le n$ and a set of orthonormal eigenvectors $\{u_{i,j}\}$ s.t.

(1) $a u_{i,j} = \lambda_i u_{i,j}$, (2) $\{u_{i,j}\}$ is an o-N basis of $\mathbb{C}^n$.

Let $\mathcal{B}(\mathbb{C})$ denote the Borel σ-algebra of $\mathbb{C}$. Define $E: \mathcal{B}(\mathbb{C}) \to M_n$



s.t. $E(S) = \sum_{i : \lambda_i \in S} P_i$

where $P_i = \sum_j u_{i,j} u_{i,j}^\dagger$ Hermitian conjugate

projection onto eigenspace of a corresponding to λ_i

Then E is a projection-valued measure.

Moreover, we have $a = \sum_{i=1}^{m} \lambda_i P_i \equiv U \Lambda U^*$

$\begin{pmatrix} u_{i,j}^{\dagger} \end{pmatrix}$ $\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{pmatrix}$

This is an example of a spectral integral,

$$a = \int_{\mathbb{C}} \lambda \, dE(\lambda)$$

Classical statistics	
Events	$\sim$ characteristic functions $\Updownarrow$ projections on abelian algebras
Quantum theory	
"quantum events"	$\sim$ projections on non-abelian algebras

Projection-valued measures

Proposition 3.54.

Let (X, Σ) be a measurable space, H a Hilbert space, and $E : \Sigma \to B(H)$ a projection-valued map such that $E(X) = I$. Then, the following are equivalent:

1. *E is a PVM.*
2. *For every countable collection $\{S_0, S_1, \ldots\}$ of pairwise-disjoint sets $S_j \in E$, $\sum_{j=0}^{J} E_j$ converges as $J \to \infty$ in the weak operator topology.*
3. *For any two disjoint sets S and T, $E(S)E(T) = 0$.*

Projection-valued measures

Given a PVM $E : \Sigma \to B(H)$ and elements $\eta, \xi \in H$ we have:

- $E_{\eta,\xi} : \Sigma \to \mathbb{C}$ with $E_{\eta,\xi}(S) = \langle \eta, E(S)\xi \rangle$ is a finite complex measure.
- $E_\eta : \Sigma \to \mathbb{R}$ with $E_\eta(S) = E_{\eta,\eta}(S) = \langle \eta, E(S)\eta \rangle$ is a probability measure.

$$\langle \eta, E(S)\eta \rangle^* = \langle E(S)\eta, \eta \rangle = \langle \eta, (E(S))^* \eta \rangle = \langle \eta, E(S)\eta \rangle \Rightarrow \langle \eta, E(S)\eta \rangle \in \mathbb{R}$$

Spectral integrals

Theorem 3.55.
Given a PVM $E : \mathcal{B}(\mathbb{C}) \to B(H)$ and a bounded Borel-measurable function $f : \mathbb{C} \to \mathbb{C}$, there exists a unique operator $a \in B(H)$ such that

$$\langle \eta, a\xi \rangle = \int_{\mathbb{C}} f(\lambda)\, dE_{\eta,\xi}(\lambda).$$

Symbolically, we write

$$a = E(f) = \int_{\mathbb{C}} f(\lambda)\, dE(\lambda).$$

Spectral theorem

$a^*a = aa^*$

Theorem 3.56.
Let $a \in B(H)$ be a normal operator. Then, there exists a unique PVM $E : \mathcal{B}(\mathbb{C}) \to B(H)$, supported on the spectrum $\sigma(a) \subset \mathbb{C}$ such that

$$a = \int_{\mathbb{C}} \lambda \, dE(\lambda).$$

Remark.
If $f : \mathbb{C} \to \mathbb{C}$ is continuous on $\sigma(a)$, then $E(f)$ is identical to $f(a)$ as defined via the continuous functional calculus.

W^*-algebras

Examples: $\mathcal{A} = L^\infty(\mu)$, $\mathcal{A} = (L^1(\mu))^*$ — abelian
$\mathcal{A} = \mathcal{B}(H)$, $\mathcal{A} = (\mathcal{B}_1(H))^*$ — non-abelian

Definition 3.57.
A W^*-algebra (or abstract von Neumann algebra) $\mathcal{A}$ is a C^*-algebra that has a predual as a Banach space, i.e., we have $A = (\mathcal{A}_*)^*$ for a Banach space $\mathcal{A}_*$.

In addition to the norm and weak topologies, a W^*-algebra has the weak-* topology induced from the predual.

↪ A sequence $a_1, a_2, \ldots \in \mathcal{A}$ converges to $a \in \mathcal{A}$ in weak-* sense, if for every $\beta \in \mathcal{A}_*$ $\lim_{n\to\infty} a_n(\beta) = a(\beta)$

Definition 3.58.

- A linear map $T : \mathcal{A} \to \mathcal{B}$ between W^*-algebras $\mathcal{A}, \mathcal{B}$ is said to be normal if it is weak-* continuous.
- Correspondingly, a state $\varphi : \mathcal{A} \to \mathbb{C}$ of a W^*-algebra is called normal if there is $\rho \in \mathcal{A}_*$ such that

$$\varphi a = a\rho, \quad \forall a \in \mathcal{A}.$$

↪ $\mathcal{B}_1(H)$: trace-class operators on H
$\mathcal{B}_1(H) = \{ a \in \mathcal{B}(H) : \operatorname{tr} a \text{ is finite} \}$, equipped with the norm $\|a\|_1 = \operatorname{tr}|a|$
$|a| = \sqrt{a^* a}$

Examples of normal states:

$\mathcal{A} = L^\infty(\mu)$. Given a probability density $p \in L^1(\mu)$, i.e., $p \geq 0$, $\int p \, d\mu = 1$, we have a normal state $\varphi_p : \mathcal{A} \to \mathbb{C}$ where $\varphi_p f = \int f p \, d\mu$

All normal states of $L^\infty(\mu)$ are of this form.

$\mathcal{A} = B(H)$. Given a density operator $\rho \in B_1(H)$, i.e., $\rho \geq 0$, $\operatorname{tr}\rho = 1$, we have a normal state $\varphi_\rho : \mathcal{A} \to \mathbb{C}$, where $\varphi_\rho a = \operatorname{tr}(\rho a)$.

All normal states of $B(H)$ are of this form.

(Result: If $a \in B(H)$, $b \in B_1(H)$ then $ab \in B_1(H)$.

If $f \in L^\infty(\mu)$, $g \in L^1(\mu)$ then $fg \in L^1(\mu)$)

Commutants

$\mathcal{A} = M_n(\mathbb{C})$
$X = \mathcal{A}, \quad X' = \{cI,\ c \in \mathbb{C}\}$

Definition 3.59.
Let $\mathcal{A}$ be an algebra. The commutant of a set $X \subseteq \mathcal{A}$, denoted as X', is the set elements of $\mathcal{A}$ that commute with every element of X, i.e.,

$$X' = \{a \in \mathcal{A} : \ ax = xa, \forall x \in X\}.$$

The bicommutant of X, denoted as X'', is the commutant of X'.

Proposition 3.60.
With notation as above, the following hold.

- *X' is a subalgebra of $\mathcal{A}$.*
- *If $\mathcal{A}$ is unital, then X' is unital.*
- *If $\mathcal{A}$ is a *-algebra, then X' is a *-algebra.*
- *$X \subseteq X''$.*
- *$X''' = X'$.*

W^*-algebras

Theorem 3.61.
The set of projections of a W^*-algebra $\mathcal{A}$ spans a norm-dense subspace of $\mathcal{A}$.

$\mathcal{A} = L^\infty(\mu)$, $\pi : L^\infty(\mu) \to \mathcal{B}(L^2(\mu))$

$\pi f = M_f$, $M_f\, g = fg$, $\forall g \in L^2(\mu)$

If $L^2(\mu)$ is separable, the $L^\infty(\mu)$ is separable

Definition 3.62.
A W^*-algebra is said to be separable if it admits a faithful, normal representation on a separable Hilbert space H.

Proposition 3.63.
If a the predual $\mathcal{A}_*$ of a W^*-algebra $\mathcal{A}$ is separable in the norm topology, then $\mathcal{A}$ is separable.

Proposition 3.64.
If a W^*-algebra is infinite-dimensional, then it is non-separable in the norm topology.

Von Neumann algebras

Definition 3.65.
Let H be a Hilbert space. A (concrete) von Neumann algebra $\mathcal{A}$ is a *-subalgebra of $B(H)$ which is closed in the weak operator topology.

↳ If $a_1, a_2, \ldots \in \mathcal{A}$ and for every $\eta, \xi \in H$ $\lim_{n\to\infty} \langle \eta, a_n \xi \rangle = \langle \eta, a\xi \rangle$ for $a \in B(H)$, then $a \in \mathcal{A}$.

Theorem 3.66 (von Neumann).
Let H be a Hilbert space and M a unital *-subalgebra of $B(H)$. Then, the following are equivalent:

1. *M is a von Neumann algebra.*
2. *M is closed in the strong operator topology.*
3. *$M = M''$.*

Examples and non-examples of von-Neumann algebras

(i) $A = B(H)$

(ii) (Non-example) $A = K(H)$ (compact operators on H:

$$K(H) = \{ a \in B(H) : a = \lim_{n \to \infty} a_n \, , \, a_n \text{ finite rank} \}$$

norm topology

↳ closed in norm topology but not closed in the weak operator topology.

e.g. if $\{\phi_0, \phi_1, \ldots\}$ is an ON-basis of H, then $a_n = \mathrm{proj}\{\phi_0, \phi_1, \ldots, \phi_{n-1}\}$ lie in $K(H)$ but converge weakly to the identity which does not lie in $K(H)$ if H is ∞-dimensional.

(iii) $H = L^2(\mu)$, $A = \{$ multiplication operators by functions in $L^\infty(\mu) \}$.

Von Neumann algebras

Theorem 3.67 (Sakai).

Every von Neumann algebra has a predual, and is thus a W^*-algebra. Moreover, the predual is unique up to isometric isomorphism.

Theorem 3.68.

Every abelian von Neumann algebra is isometrically isomorphic to $L^\infty(\mu)$ for some measure space (X, Σ, μ).

Analogously to our interpretation of the study of C^*-algebras as "non-commutative topology", we can interpret the study of von Neumann algebras as "non-commutative measure theory".