Section 4

Embedding dynamical systems in operator algebras

Dirac–von Neumann axioms of quantum mechanics

1. States are density operators, i.e., positive, trace-class operators $\rho : H \to H$ on a Hilbert space H, with $\operatorname{tr} \rho = 1$.
2. Observables are self-adjoint operators, $A : D(A) \to H$.
   $\subseteq H$, domain of A
3. Measurement expectation and probability:

$$\mathbb{E}_\rho A = \operatorname{tr}(\rho A), \quad \mathbb{P}_\rho(\Omega) = \mathbb{E}_\rho(E(\Omega)), \quad A = \int_{\mathbb{R}} a \, dE(a).$$

$E : \mathcal{B}(\mathbb{R}) \to \mathcal{B}(H)$
PVM

4. Unitary dynamics between measurements:

$$\rho_t = U^{t*} \rho_0 U^t.$$

$U^t : H \to H$
unitary

5. Projective measurement:

$$\rho|_e = \frac{\sqrt{e}\rho\sqrt{e}}{\operatorname{tr}(\sqrt{e}\rho\sqrt{e})}, \quad 0 < e \leq I.$$

quantum effect
"non abelian fuzzy event"

Algebraic formulation: States and observables

1. Associated with a physical system is a unital C^*-algebra $\mathcal{A}$.
2. The set of states of the system is the state space $S(\mathcal{A})$ of $\mathcal{A}$.
3. The set of observables of the system is the set of self-adjoint elements $\mathcal{A}_{\mathrm{sa}}$ of $\mathcal{A}$.
4. The set of values that can be obtained in a measurement of $a \in A_{\mathrm{sa}}$ corresponds to the spectrum $\sigma(a) \subset \mathbb{R}$.
5. The expected value of a measurement of $a \in \mathcal{A}_{\mathrm{sa}}$ when the system is in state $\varphi \in S(\mathcal{A})$ is given by $\varphi(a)$.

Abelian

$\mathcal{A} = L^{\infty}(\mu)$

$\mathcal{A}_{sa} = \{ f \in \mathcal{A} \mid \text{real-valued} \}$

$\sigma(f)$ = essential range of f

Non-abelian

$\mathcal{A} = B(H)$

$\mathcal{A}_{sa} = \{ a \in B(H) \mid \langle \eta, a\xi \rangle = \langle a\eta, \xi \rangle \}$

$\sigma(a)$ = spectrum of operator

Algebraic formulation: Events and measurement probabilities

- The set of events (or effects) that can be observed is the set of positive elements $e \in \mathcal{A}_+$ such that $0 \leq e \leq \mathbb{1}$. If the system is in state $\varphi \in S(\mathcal{A})$, the probability to observe e is given by $\varphi(e)$.
- Supposing, further, that $\mathcal{A}$ is a W^*-algebra, the measurement probability for a to take value in a set $S \in \mathcal{B}(\mathbb{R})$ is given by $\varphi(E(S))$, where $E : \mathcal{B}(\mathbb{R}) \to \mathcal{A}$ is the PVM satisfying $a = \int_{\mathbb{R}} \lambda \, dE(\lambda)$.

Completely positive maps

e.g. $M_2(\mathcal{A})$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{pmatrix}$$

$a_{ij} \in \mathcal{A}$, $b_{ij} \in \mathcal{A}$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^* = \begin{pmatrix} a_{11}^* & a_{21}^* \\ a_{12}^* & a_{22}^* \end{pmatrix}$$

Notation.
Given a C^*-algebra $\mathcal{A}$, $\mathbb{M}_n(\mathcal{A})$ is the C^*-algebra of $n \times n$ matrices with entries in $\mathcal{A}$.

Definition 4.1.
Let $T : \mathcal{A} \to \mathcal{B}$ be a linear map between C^*-algebras $\mathcal{A}$ and $\mathcal{B}$. Given $n \in \mathbb{N}$, we say that the map $T^{(n)} : \mathbb{M}_n(\mathcal{A}) \to \mathbb{M}_n(\mathcal{B})$ defined as $T^{(n)}([a_{ij}]) = [T(a_{ij})]$ is a matrix amplification of T.

Recall, T positive $\Leftrightarrow$ $T(a) \geq 0$ whenever $a \geq 0$ $\Leftrightarrow$ $a = c^*c$

Definition 4.2.
A linear map $T : \mathcal{A} \to \mathcal{B}$ between C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ is said to be:

- n-positive if $T^{(n)}$ is positive.
- Completely positive if it is n-positive for every $n \in \mathbb{N}$.

$$M_n(\mathbb{C}) \otimes \mathcal{A} \xrightarrow{I \otimes T} M_n(\mathbb{C}) \otimes \mathcal{B}$$

$\simeq M_n(\mathcal{A})$ $\quad$ $\simeq T^{(n)}$

Example of a positive map which is not completely positive

$\mathcal{A} = M_2(\mathbb{C})$, $T : \mathcal{A} \to \mathcal{A}$, $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ (matrix transpose)

For any $x \in \mathbb{C}^2$, $a \in \mathcal{A}_+$, we have $\langle x, T(a)x \rangle = \langle x, (a^* x^*)^* \rangle = \langle x^*, a^* x^* \rangle^*$

$$= \underbrace{\langle a x^*, x^* \rangle}_{\geq 0 \text{ since } a \in \mathcal{A}_+}{}^* \geq 0 \implies T \text{ is positive.}$$

Claim: T is not 2-positive. Indeed,

$$\underbrace{T^{(2)}\left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right)}_{\text{positive as an element of } M_2(\mathcal{A})} = \underbrace{\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)}_{\text{not positive since the determinant is } = -1} \implies T \text{ is not 2-positive.}$$

Completely positive maps

Theorem 4.3 (Stinespring).

Let $\mathcal{A}$ be a C^*-algebra and H a Hilbert space. A linear map $T : \mathcal{A} \to B(H)$ is completely positive iff there is a Hilbert space K, a representation $\pi : \mathcal{A} \to B(K)$ and a bounded linear map $V : K \to H$ such that

$$Ta = V(\pi a)V^*, \quad \forall a \in \mathcal{A}.$$

Proposition 4.4.

With notation as above, if $\mathcal{A}$ is abelian then $T : \mathcal{A} \to B(H)$ is completely positive iff it is positive.

Theorem 4.5 (Choi).

Let K and H be finite-dimensional Hilbert spaces of dimension m and n, respectively. Then, any completely positive map $T : B(K) \to B(H)$ take s the form $T(a) = \sum_{i=1}^{mn} V_i a V_i^*$ for some operators $V_i : K \to H$.

↳ Kraus operators

Quantum operations, quantum channels

Definition 4.6.
A linear map $T : \mathcal{B} \to \mathcal{A}$ between unital C^*-algebras $\mathcal{B}$ and $\mathcal{A}$ is said to be a quantum operation if:

1. T is completely positive.
2. $T\mathbb{1}_{\mathcal{B}} \leq \mathbb{1}_{\mathcal{A}}$.

If $T\mathbb{1}_{\mathcal{B}} = \mathbb{1}_{\mathcal{A}}$, T is said to be a quantum channel.

Proposition 4.7.
If $T : \mathcal{B} \to \mathcal{A}$ is a quantum operation, then for every state $\omega \in S(\mathcal{A})$ $T^*\omega \in \mathcal{B}^*$ is a positive functional satisfying $(T^*\omega)\mathbb{1}_B \leq 1$. Moreover, if T is a quantum channel, $(T^*\omega)\mathbb{1}_B = 1$.

Corollary 4.8.
The adjoint $T^* : \mathcal{A}^* \to \mathcal{B}^*$ of a quantum channel $T : \mathcal{B} \to \mathcal{A}$ maps the state space $S(\mathcal{A})$ into $S(\mathcal{B})$.

$T^*\varphi = \varphi \circ T$

Quantum operations, quantum channels

Proposition 4.9.

A normal (weak-* continuous) quantum operation $T : \mathcal{B} \to \mathcal{A}$ between W^*-algebras $\mathcal{B}$ and $\mathcal{A}$ has a predual, i.e., $T = (T_*)^*$ for a unique linear map $T_* : \mathcal{A}_* \to \mathcal{B}_*$.

Algebraic formulation of measure-preserving dynamics

State space dynamics

$$\Phi : \Omega \to \Omega$$
$$\Phi_* : \mathcal{M}(\Omega) \to \mathcal{M}(\Omega), \quad \Phi_* \alpha = \alpha \circ \Phi^{-1}$$

- Φ: Invertible measure-preserving map.
- $\mathcal{M}$: Space of Borel measures on Ω.
- Φ_*: Pushforward map on measures.
- μ: Invariant probability measure, $\Phi_* \mu = \mu$.

Algebraic formulation of measure-preserving dynamics

Abelian formulation

$$U : \mathcal{A} \to \mathcal{A}, \quad Uf = f \circ \Phi$$
$$P : S_*(\mathcal{A}) \to S_*(\mathcal{A}), \quad Pp = p \circ \Phi^{-1}$$

- $\mathcal{A} = L^\infty(\mu)$: Abelian von Neumann algebra.
- $\mathcal{A}_{\text{sa}} = \{f \in \mathcal{A} : f \text{ is real-valued}\}$: Classical observables.
- $U : \mathcal{A} \to \mathcal{A}$: Koopman operator.
- $\mathcal{A}_* = L^1(\mu)$: Predual.
- $S_*(\mathcal{A}) = \{p \in \mathcal{A}_* : p \geq 0, \int_\Omega p \, d\mu = 1\}$: Probability densities.
- $\mathbb{E}_p : \mathcal{A} \to \mathbb{C}$ with $p \in S_*(\mathcal{A})$: Normal states, $\mathbb{E}_p f = \int_\Omega fp \, d\mu$.
- $P : S_*(\mathcal{A}) \to S_*(\mathcal{A})$: Transfer operator.

Algebraic formulation of measure-preserving dynamics

Non-abelian formulation

$$\mathcal{U} : \mathcal{B} \to \mathcal{B}, \quad \mathcal{U} a = U a U^*$$
$$\mathcal{P} : \mathcal{S}_*(\mathcal{B}) \to S_*(\mathcal{B}), \quad \mathcal{B}\rho = U^* \rho U$$

- $H = L^2(\mu)$: Hilbert space.
- $U : H \to H$: Unitary Koopman operator, $Uf = f \circ \Phi$.
- $\mathcal{B} = B(H)$: Non-abelian von Neumann algebra.
- $\mathcal{B}_{\text{sa}} = \{a \in \mathcal{B} : \ a \text{ is self-adjoint}\}$: Quantum observables.
- $\mathcal{U} : \mathcal{B} \to \mathcal{B}$: Induced Koopman operator.
- $\mathcal{B}_* = B_1(H)$: Predual.
- $S_*(\mathcal{B}) = \{\rho \in \mathcal{B}_* : \ \rho \geq 0, \operatorname{tr} \rho = 1\}$: Density operators.
- $\mathbb{E}_\rho : \mathcal{B} \to \mathbb{C}$ with $\rho \in \mathcal{S}_*(\mathcal{B})$: Normal states, $\mathbb{E}_\rho a = \operatorname{tr}(a\rho)$.
- $\mathcal{P} : S_*(\mathcal{B}) \to S_*(\mathcal{B})$: Induced transfer operator.

Algebraic formulation of measure-preserving dynamics

Classical–quantum consistency

Proposition 4.10.
The maps $U : \mathcal{A} \to \mathcal{A}$ and $\mathcal{U} : \mathcal{B} \to \mathcal{B}$ are quantum channels. Moreover, the following diagrams commute for the injective maps $\pi : \mathcal{A} \to \mathcal{B}$ and $\Gamma : S_*(\mathcal{A}) \to S_*(\mathcal{B})$:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "U"] \arrow[d, "\pi"'] & \mathcal{A} \arrow[d, "\pi"] \\ \mathcal{B} \arrow[r, "\mathcal{U}"] & \mathcal{B} \end{tikzcd} \qquad \begin{tikzcd} S_*(\mathcal{A}) \arrow[r, "P"] \arrow[d, "\Gamma"'] & S_*(\mathcal{A}) \arrow[d, "\Gamma"] \\ S_*(\mathcal{B}) \arrow[r, "\mathcal{P}"] & S_*(\mathcal{B}) \end{tikzcd}$$

- $\pi : \mathcal{A} \to \mathcal{B}$: Regular representation, $\pi f = a$ with $ag = fg$ for all $g \in H$.
- $\Gamma : S_*(\mathcal{A}) \to S_*(\mathcal{B})$: Mapping of probability densities into pure quantum states, $\Gamma(\pi) = \langle \sqrt{p}, \cdot \rangle \sqrt{p}$.

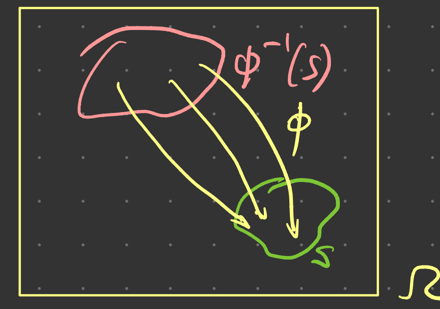
ALGEBRAIC FORMULATION OF MEASURE-PRESERVING DYNAMICAL SYSTEMS

- State space dynamics:

$\Phi : \Omega \to \Omega$ (Ω, standard Borel measurable space)

$\Phi_* : \mathcal{M}(\Omega) \to \mathcal{M}(\Omega)$, $(\Phi_* \nu)(S) = \nu(\Phi^{-1}(S))$

$\Phi_* \mu = \mu$ (invariant probability measure)



	Abelian	Non-abelian
von Neumann algebra	$\mathcal{A} = L^\infty(\mu)$	$\mathcal{B} = B(H)$, $H = L^2(\mu)$
observables	$\mathcal{A}_{sa} = \{$real-valued elements of $\mathcal{A}\}$	$\mathcal{B}_{sa} = \{$self-adjoint linear maps in $\mathcal{B}\}$
predual	$\mathcal{A}_* = L^1(\mu)$	$\mathcal{B}_* = B_1(H)$
normal states	$S_*(\mathcal{A}) = \{p \in L^1(\mu) : p \geq 0, \int p \, d\mu = 1\}$ $\mathbb{E}_p f = \int f p \, d\mu$, $p \in S_*(\mathcal{A})$, $f \in \mathcal{A}$	$S_*(\mathcal{B}) = \{\rho \in B_1(H) : \rho \geq 0, \operatorname{tr} \rho = 1\}$ $\mathbb{E}_\rho a = \operatorname{tr}(\rho a)$, $\rho \in S_*(\mathcal{B})$, $a \in \mathcal{B}$
Quantum channel	$U : \mathcal{A} \to \mathcal{A}$, $Uf = f \circ \Phi$ ($f \geq 0 \Rightarrow Uf \geq 0$ U is c.p. since $\mathcal{A}$ is abelian)	$\mathcal{U} : \mathcal{B} \to \mathcal{B}$, $\mathcal{U} a = U a U^*$ ($U : H \to H$, unitary Koopman op. $Uf = f \circ \Phi$. $\mathcal{U}$ is cp. by Stinespring thm)
Predual	$P \equiv U_* : \mathcal{A}_* \to \mathcal{A}_*$, $P \alpha = \alpha \circ \Phi^{-1}$	$P = \mathcal{U}_* : \mathcal{B}_* \to \mathcal{B}_*$, $P \rho = U^* \rho U$

EMBEDDING CLASSICAL (ABELIAN) DYNAMICS INTO QUANTUM (NON-ABELIAN) DYNAMICS

Observables

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{U} & \mathcal{A} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \mathcal{B} & \xrightarrow{\mathcal{U}} & \mathcal{B} \end{array}$$

Regular rep:

$\pi : \mathcal{A} \to \mathcal{B}$

$\pi f = a$ where $a g = f g$

forall $g \in H$

$$\mathcal{U}(\pi f)\, g = U(\pi f) U^* g$$
$$= U(f U^* g)$$
$$= (Uf)\, U U^* g$$
$$= Uf\, g = \pi(Uf)\, g$$

$$\Rightarrow \boxed{\mathcal{U} \circ \pi = \pi \circ U}$$

(Normal) states

$$\begin{array}{ccc} S_*(\mathcal{A}) & \xrightarrow{P} & S_*(\mathcal{A}) \\ \downarrow{\scriptstyle \Gamma} & & \downarrow{\scriptstyle \Gamma} \\ S_*(\mathcal{B}) & \xrightarrow{\mathcal{P}} & S_*(\mathcal{B}) \end{array}$$

Embedding of prob. densities

$\Gamma : S_*(\mathcal{A}) \to S_*(\mathcal{B})$

$\Gamma p = \rho \;,\quad \rho g = \langle \sqrt{p}, g \rangle \sqrt{p}$

for all $g \in H$

(Pure state)

$$\mathcal{P}(\Gamma_p)\, g = U^*(\Gamma_p) U g = U^* \langle \sqrt{p}, U g \rangle \sqrt{p}$$
$$= \langle \sqrt{p}, U g \rangle U^* \sqrt{p} = \langle U^* \sqrt{p}, g \rangle U^* \sqrt{p}$$
$$= \langle \sqrt{U^* p}, g \rangle \sqrt{U^* p} = \langle \sqrt{P_p}, g \rangle \sqrt{P_p} = \Gamma(P_p)$$

$$\Rightarrow \boxed{\mathcal{P} \circ \Gamma = \Gamma \circ P}$$

Classical–quantum consistency: $\forall f \in \mathcal{A},\ p \in S_*(\mathcal{A})$, $\mathbb{E}_{P_p} f = \mathbb{E}_{\mathcal{P}(\Gamma_p)} (\pi f) = \mathbb{E}_{\Gamma_p}(\mathcal{U}(\pi f))$

$\mathbb{E}_{P_p} f = \mathbb{E}_p(Uf)$

Heisenberg

Schrodinger

Heisenberg

TOEPLITZ MATRIX APPROXIMATION OF MULTIPLICATION OPERATORS

Given $f \in \mathcal{A} = L^\infty(\mu)$, compute $A \in \mathbb{M}_{2\ell+1}$ s.t.

$$A_{ij} = \langle \phi_i, (\pi f)\phi_j \rangle = \langle \phi_i, f \phi_j \rangle \qquad i,j \in -2\ell, \dots, 2\ell$$

$$f = \sum_{k \in \mathbb{Z}} \hat{f}_k \phi_k \Rightarrow A_{ij} = \langle \phi_i, \sum_k \hat{f}_k \phi_k \phi_j \rangle = \sum_{k \in \mathbb{Z}} \langle \phi_i, \phi_j \phi_k \rangle \hat{f}_k$$

$$= \sum_{k \in \mathbb{Z}} \langle \phi_i, \phi_{j+k} \rangle \hat{f}_k = \sum_{k \in \mathbb{Z}} \delta_{i,j+k} \hat{f}_k = \hat{f}_{i-j}$$

Example: $\ell = 2$

$$A = \begin{pmatrix} \hat{f}_0 & \hat{f}_{-1} & \hat{f}_{-2} & \hat{f}_{-3} & \hat{f}_{-4} \\ \hat{f}_1 & \ddots & \ddots & \ddots & \\ \hat{f}_2 & \ddots & \ddots & \ddots & \\ \hat{f}_3 & \ddots & \ddots & & \\ \hat{f}_4 & & & & \end{pmatrix}$$

$$A u_j = a_j u_j \qquad u_j^\dagger u_k = \delta_{jk} \quad (a_j \geq 0 \text{ whenever } f \geq 0).$$

$u_j = (u_{-\ell j}, \dots, u_{\ell j})^T \in \mathbb{C}^{2\ell+1}$ — stores expansion coefficients of eigenfunctions of the operator represented by A

$$\psi_j = \sum_{k=-\ell}^{\ell} u_{kj} \phi_j$$

eigenfunction

FINITE-DIMENSIONAL APPROXIMATION

Observables

$$\begin{tikzcd}
\mathfrak{B} \arrow[r, "\mathcal{U}"] \arrow[d, "\tilde{\Pi}_L"'] & \mathfrak{B} \arrow[d, dashed, "\tilde{\Pi}_L"] \\
\mathfrak{B}_L \arrow[r, "\mathcal{U}_L"] & \mathfrak{B}_L
\end{tikzcd}$$

States

$$\begin{tikzcd}
S_*(\mathfrak{B}) \arrow[r, "\mathcal{P}"] \arrow[d, "\tilde{\Pi}_L'"'] & S_*(\mathfrak{B}) \arrow[d, dashed, "\tilde{\Pi}_L'"] \\
S_*(\mathfrak{B}_L) \arrow[r, "\mathcal{P}_L"] & S_*(\mathfrak{B}_L)
\end{tikzcd}$$

- $H_1 \subset H_2 \subset \dots \subset H$, H_L finite-dimensional subspace (e.g. $H_L = \{\phi_0, \dots, \phi_{L-1}\}$ for an o-n basis $\{\phi_0, \phi_1, \dots\}$ of H.
- Projections: $\Pi_L : H \to H$, $\operatorname{ran} \Pi_L = H_L$

(see assignment 2)

- $\tilde{\Pi}_L : \mathfrak{B} \to \mathfrak{B}_L = \{\Pi_L a \Pi_L : a \in B(H)\} \simeq B(H_L)$
- $\mathcal{U}_L : \mathfrak{B}_L \to \mathfrak{B}_L$, $\mathcal{U}_L a = U_L a U_L^*$, $U_L = \Pi_L U \Pi_L$ — projected Koopman op

In general, U_L is not unitary, i.e., $U_L^* \neq U_L^{-1}$, unless $\Pi_L U = U \Pi_L$. As a result, $\mathcal{U}_L$ is not, in general a quantum channel (in particular, $\mathcal{U}_L \mathbb{1}_{\mathfrak{B}_L} \neq \mathbb{1}_{\mathfrak{B}_L}$). However, it is a quantum operation, i.e. $\mathcal{U}_L$ is completely positive (by Stinespring thm.) and $\mathcal{U}_L \mathbb{1}_{\mathfrak{B}_L} \leq \mathbb{1}_{\mathfrak{B}_L}$.

- Moreover the diagrams do not in general commute (unless $\Pi_L U = U \Pi_L$)
- Projected states: $\tilde{\Pi}_L'(\rho) = \dfrac{\Pi_L \rho \Pi_L}{tr(\Pi_L \rho \Pi_L)}$. Projected transfer operator: $\mathcal{P}_L \rho = \dfrac{U_L^* \rho U_L}{tr(U_L^* \rho U_L)}$
- Projected multiplication operators: $\pi_L : \mathfrak{A} \to \mathfrak{B}_L$, $\pi_L = \tilde{\Pi}_L \circ \pi$, $\pi_L f = \Pi_L \pi(f) \Pi_L$

↪ (see assignment 3)

Remarks

(i) $\pi_L f$ is not a multiplication operator by an element in $\mathcal{A}$ e.g. $\Pi_L f$

(ii) $\Gamma_L p \equiv \Pi_L'(\Gamma p)$ is not a state induced by a probability density in $S_*(\mathcal{A})$

RECONSTRUCTING CLASSICAL OBSERVABLES

<u>Goal:</u> Given $A_L = \pi_L f \equiv \Pi_L(\pi f)\Pi_L$ projected multiplication operator reconstruct a function $f_L \in \mathcal{A}$ s.t. f_L approximates f in some sense.

$f_L : \Omega \to \mathbb{C}$

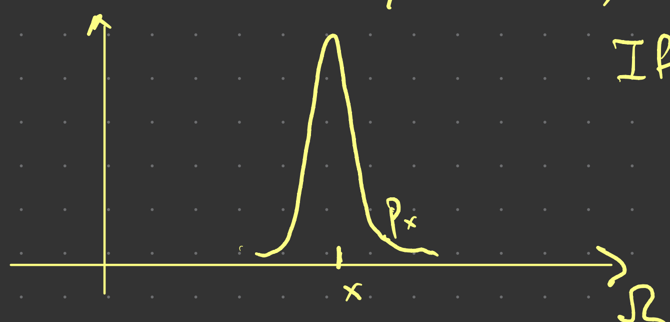
Observe that starting from the C^*-algebra $C(\Omega) \subseteq \mathcal{A}$ we have the characters $\delta_x \in S(C(\Omega))$, $x \in \Omega$, s.t. $\delta_x f = f(x)$. There exists an extension $\tilde{\delta}_x \in S(\mathcal{A})$ i.e. $\tilde{\delta}_x f = \delta_x f$ for all $f \in C(\Omega)$. Given $f \in \mathcal{A}$ this motivates defining $\tilde{f} : \Omega \to \mathbb{C}$ s.t. $\tilde{f}(x) = \tilde{\delta}_x f$. However, $\tilde{\delta}_x$ is not a normal state, so we cannot use the map $\Gamma : S_*(\mathcal{A}) \to S_*(\mathcal{B})$ to represent it quantum mechanically.

<u>Approach:</u> Replace $\tilde{\delta}_x$ by a normal state $p_x \in S_*(\mathcal{A})$ that approximates pointwise evaluation.

Consider $p : \Omega \times \Omega \to \mathbb{R}$ continuous <u>Markov kernel</u>:

(i) $p \geq 0$

(ii) For every $x \in \Omega$, $p_x \equiv p(x, \cdot)$ is a probability density in $S_*(\mathcal{A})$



If p_x is "concentrated" around x, we have

$$\mathbb{E}_{p_x} f \approx f(x) = \delta_x f \quad \text{for } f \in C(\Omega)$$

Example: $\Omega = S^1$, $p : S^1 \times S^1 \longrightarrow \mathbb{R}$ $\quad p(x,x') = \frac{1}{I_0(1/\varepsilon^2)} \exp\left(\frac{\cos(x-x')}{\varepsilon^2} \right)$
(Von-Mises kernel)

With any such choice $p : \Omega \times \Omega \to \mathbb{R}$, define $R : \mathcal{B} \longrightarrow C(\Omega)$ s.t.

$$g = Ra \quad , \quad g(x) = \mathbb{E}_{\Gamma(p_x)} a = \operatorname{tr}(\Gamma(p_x)\, a) = \langle \sqrt{p_x}, a \sqrt{p_x} \rangle$$

If $a = \pi f$ for some $f \in \mathcal{A}$, we have $g(x) = \mathbb{E}_{p_x} f \simeq f(x)$

Finite-dimensional approximation: $R_L : \mathcal{B}_L \longrightarrow C(\Omega)$ s.t.

$$g_L = R_L a \;, \quad g_L(x) = \mathbb{E}_{\Pi_L'(\Gamma(p_x))} = \operatorname{tr}\left(\Pi_L'(\Gamma(p_x))\, a \right) = \frac{\langle \Pi_L \sqrt{p_x}, a\, \Pi_L \sqrt{p_x} \rangle}{\| \Pi_L \sqrt{p_x} \|_H^2}$$

Given a basis $\{\phi_0, \dots, \phi_{L-1}\}$ for H_L $\quad a$ is represented by matrix $A_{ij} = \langle \phi_i, a \phi_j \rangle$

$\xi_x = \frac{\Pi_L \sqrt{p_x}}{\|\Pi_L \sqrt{p_x}\|}$ is represented by vector $v_j = \langle \phi_j, \xi_x \rangle$

$$\Rightarrow \quad g_L(x) = v^\dagger A v$$

ASYMPTOTIC CONSISTENCY ($L \to \infty$)

Prop (a) For any density operator $\rho \in S_*(\mathcal{B})$, the projected density operators $\rho_L = \Pi_L'(\rho) = \dfrac{\Pi_L \rho \Pi_L}{tr(\Pi_L \rho \Pi_L)}$ converge to ρ in the trace norm of $S_*(\mathcal{B})$,

i.e. $\lim\limits_{L\to\infty} \|\rho - \rho_L\|_1 = 0$ where $\|a\|_1 = tr|a|$, $|a| = \sqrt{a^* a}$, $a \in B_1(H)$.

(b) For any bounded operator $a \in \mathcal{B}$, $a_L = \Pi_L a \Pi_L$ converges to a in the strong topology of $\mathcal{B}$, i.e., for every $f \in H$, $\lim\limits_{L\to\infty} a_L f = af$.

Corollary: For any $a \in \mathcal{B}$, $\rho \in S_*(\mathcal{B})$, $\lim\limits_{L\to\infty} \mathbb{E}_{\rho_L} a_L = \mathbb{E}_\rho a$

Pf. Observe $\mathbb{E}_{\rho_L} a_L \equiv tr(\rho_L a_L) = tr(\rho_L \Pi_L a \Pi_L) = tr(\Pi_L \rho_L \Pi_L a) = tr(\rho_L a)$

Thus, $|\mathbb{E}_{\rho_L} a_L - \mathbb{E}_\rho a| = |tr((\rho_L - \rho)a)| \le tr|(\rho_L - \rho)a| \le \|\rho_L - \rho\|_1 \|a\|$

$\Rightarrow |\mathbb{E}_{\rho_L} a_L - \mathbb{E}_\rho a| \xrightarrow[L\to\infty]{} 0$.

Similarly, for $\mathcal{P} : S_*(\mathcal{B}) \to S_*(\mathcal{B})$ and $\mathcal{P}_L : S_*(\mathcal{B}_L) \longrightarrow S_*(\mathcal{B}_L)$,

we have $\lim\limits_{L\to\infty} \mathbb{E}_{(\mathcal{P}_L \rho_L)} a_L = \mathbb{E}_{\mathcal{P}\rho}\, a$.