Section 5

Spectral theory of one-parameter evolution groups

Setting and objectives

General assumptions

- $\Phi : G \times \Omega \to \Omega$: Continuous-time, continuous flow on compact, metrizable space Ω.
- μ: Ergodic invariant Borel probability measure.
- $X : \Omega \to \mathbb{X}$ continuous observation map into metric space $\mathcal{X}$.
- $U^t : \mathcal{F} \to \mathcal{F}$: Koopman operator on Banach space $\mathcal{F}$ of complex-valued observables.

Given. Time-ordered samples

$$x_n = X(\omega_n), \quad \omega_n = \Phi^{t_n}(\omega_0), \quad t_n = (n-1)\,\Delta t.$$

Goal. Using the data x_n, identify a collection of observables $\zeta_j : \Omega \to \mathcal{Y}$ which have the property of evolving coherently under the dynamics in a suitable sense.

Setting and objectives

We recall the following facts from Section 2 (see Proposition 2.7 and Theorems 2.29, 2.30).

Theorem 5.1.

1. $\{U^t : C(\Omega) \to C(\Omega)\}_{t\in\mathbb{R}}$ *is a strongly continuous group of isometries.*
2. $\{U^t : L^p(\mu) \to L^p(\mu)\}_{t\in\mathbb{R}}$, $p \in [0, \infty)$ *is a strongly continuous group of isometries. Moreover,* $U^t : L^2(\mu) \to L^2(\mu)$ *is unitary.*
3. $\{U^t : L^\infty(\mu) \to L^\infty(\mu)\}_{t\in\mathbb{R}}$ *is a weak-** *continuous group of isometries.*

Notation.

- $\mathcal{F}$: Any of the $C(\Omega)$ or $L^p(\mu)$ spaces with $1 \le p \le \infty$.
- $\mathcal{F}_0$: Any of the $C(\Omega)$ or $L^p(\mu)$ spaces with $1 \le p < \infty$.
- C_0 (semi)group $\equiv$ strongly continuous (semi)group.
- C_0^* (semi)group $\equiv$ weak-* continuous (semi)group.

Generator of C_0 semigroups

Definition 5.2.
Let $\{S^t\}_{t\geq 0}$ be a C_0 semigroup on a Banach space E. The generator $A : D(A) \to E$ of the semigroup $\{S^t\}_{t\geq 0}$ is defined as

$\subseteq E$, domain of A

$$Af = \lim_{t\to 0} \frac{S^t f - f}{t}, \quad f \in D(A),$$

where the limit is taken in the norm of E, and the domain $D(A) \subseteq E$ consists of all $f \in E$ for which the limit exists.

Example (Circle rotation)

$\phi^t : S^1 \to S^1$, $\phi^t(\theta) = (\theta + \alpha t) \bmod 2\pi$, $\alpha \in \mathbb{R}$ (frequency parameter). Consider $U^t : C(S^1) \to C(S^1)$. For $f \in C^1(S^1)$, we have

$$\lim_{t\to 0} \frac{U^t f(\theta) - f(\theta)}{t} = \lim_{t\to 0} \frac{f(\phi^t(\theta)) - f(\theta)}{t} = \lim_{t\to 0} \frac{f(\theta+\alpha t) - f(\theta)}{t} = \lim_{t\to 0} \frac{f(\theta+\alpha t) - f(\theta)}{\alpha t}\, \alpha = \alpha f'(\theta) \equiv Af(\theta)$$

Moreover,

$$\lim_{t\to 0} \left\| \frac{U^t f - f}{t} - \alpha f' \right\|_{C(S^1)} = \lim_{t\to 0} \max_{\theta \in S^1} \left| \frac{f(\theta+\alpha t) - f(\theta)}{t} - \alpha f'(\theta) \right| = 0.$$

Thus, (i) $D(A) \supseteq C^1(S^1)$; (ii) For $f \in C^1(S^1)$, $Af = \alpha f'$. In fact, in this case $D(A) = C^1(S^1)$.

More generally for a flow $\phi^t : \Omega \to \Omega$ on a manifold, where $\frac{d}{dt}(\phi^t(\omega)) = v(\phi^t(\omega))$, we have $A = \vec{v} \cdot \vec{\nabla}$

Moreover, A is an <u>unbounded operator</u>.

Recall that $A : D(A) \to E$ is bounded iff $\sup_{f \in E \setminus \{0\}} \dfrac{\|Af\|_E}{\|f\|_E} < \infty$. A is said to be unbounded if no such bound exists.

For our example, we can take the sequence $f_n \in C^1(S^1)$ with $f_n(\theta) = e^{in\theta}$, giving

$$\frac{\|Af_n\|_{C(S^1)}}{\|f_n\|_{C(S^1)}} = \|\alpha f_n'\|_{C(S^1)} = \|\alpha i n f_n\|_{C(S^1)} = |\alpha| n$$

Thus, f_n is a sequence of unit vectors in $C(S^1)$ for which $\|Af_n\|_{C(S^1)}$ increases without bound.

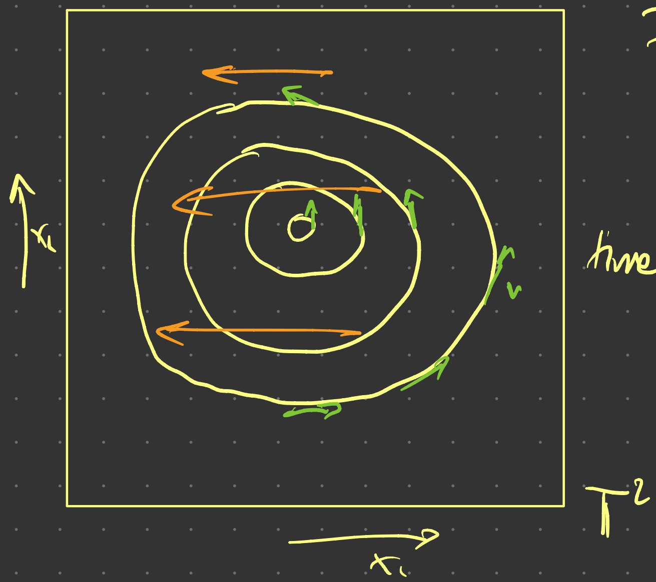
$\Rightarrow$ A is unbounded.

<u>Thm.</u> A linear operator $A : F \to E$ from a normed space F to a Banach space E is continuous iff it is bounded. That is, if $\lim_{n\to\infty} Af_n = Af$ for <u>every</u> sequence f_n s.t. $\lim_{n\to\infty} f_n = f$ then A is bounded.

<u>Def</u>. $A : D(A) \to E$ is <u>closed</u> if for every $f_n \in D(A)$ converging to $f \in E$ such that $g_n = Af_n$ converges to $g \in E$ we have

(i) $f \in D(A)$

(ii) $g = Af$.



Initial-value problem

$$\dot{\omega}(t) = v(\omega(t)) \qquad v : \mathbb{T}^2 \longrightarrow \mathbb{R}^2 \text{ vector field}$$

$$\omega(0) = \omega_0 .$$

time-t Solution map: $\phi^t : \mathbb{T}^2 \longrightarrow \mathbb{T}^2$, $\phi^t(\omega_0) = \omega(t)$

Assume: $\operatorname{div} v = \dfrac{\partial v_1}{\partial x_1} + \dfrac{\partial v_2}{\partial x_2} = 0$

The flow ϕ^t preserves the Lebesgue measure

$\Rightarrow$ $U^t : L^2(\mathbb{T}^2) \longrightarrow L^2(\mathbb{T}^2)$

$U^t f = f \circ \phi^t$ is a C_0 unitary evolution group.

Generator $V : D(V) \rightarrow L^2(\mathbb{T}^2)$. Given $f \in C^1(\mathbb{T}^2) \subset D(V)$, we have

$$Vf = \lim_{t \to 0} \frac{U^t f - f}{t} = v \cdot \nabla f$$

Generator is a directional derivative on observables.

Generator of C_0 semigroups

Theorem 5.3.
With the notation of Definition 5.2, the following hold.

1. *A is closed and densely defined.*
2. *For all $f \in D(A)$ and $t \geq 0$, the function $t \mapsto S^t f$ is continuously differentiable, and satisfies*

$$\frac{d}{dt} S^t f = A S^t f = S^t A f.$$

3. *A uniquely characterizes the semigroup $\{S^t\}$, i.e., if $\{\tilde{S}^t\}$ is another C_0 semigroup on E with the same generator A, then $S^t = \tilde{S}^t$ for all $t \geq 0$.*

Generator of C_0^* semigroups

Definition 5.4.
Let $\{S^t\}_{t\geq 0}$ be a C_0^* semigroup on a Banach space E with predual E_*. The generator $A : D(A) \to E$ of the semigroup $\{S^t\}_{t\geq 0}$ is defined as the weak-* limit

$$\langle g, Af \rangle = \lim_{t\to 0} \frac{\langle g, S^t f - f \rangle}{t}, \quad f \in D(A), \quad \forall g \in E_*,$$

where the domain $D(A) \subseteq E$ consists of all $f \in E$ for which the limit exists.

Theorem 5.5.
With the notation of Definition 5.4, the following hold.

1. *A is weak-* closed and densely defined.*
2. *For all $f \in D(A)$ and $t \geq 0$, the function $t \mapsto S^t f$ is weak-* continuously differentiable, and satisfies*

$$\left\langle g, \frac{d}{dt} S^t f \right\rangle = \langle g, AS^t f \rangle = \langle g, S^t Af \rangle.$$

3. *A uniquely characterizes the semigroup $\{S^t\}$, i.e., if $\{\tilde{S}^t\}$ is another C_0^* semigroup on E with the same generator A, then $S^t = \tilde{S}^t$ for all $t \geq 0$.*

Generator of unitary C_0 groups

Theorem 5.6 (Stone).
Let $\{S^t\}_{\cancel{t \geq 0}\; t\in\mathbb{R}}$ be a unitary C_0 group on a Hilbert space H. Then, the generator $A : D(A) \to H$ is skew-adjoint, i.e.,

$$A^* = -A.$$

Conversely, if $A : D(A) \to H$ is skew-adjoint, it is the generator of a unitary evolution group.

Generalizes the result from matrix algebra that if $A \in M_n(\mathbb{C})$ is skew-adjoint, $S^t = e^{tA}$ (defined, in this case by Taylor series) is unitary.

Note: $A^* = -A \Rightarrow \langle f, Ag \rangle = -\langle f, A^* g \rangle = -\langle Af, g \rangle \Rightarrow A$ is antisymmetric

In infinite dimensions antisymmetric $\not\Leftarrow$ skew-adjoint.

Generator of Koopman evolution groups

Corollary 5.7.

Under our general assumptions the following hold:

1. *The Koopman evolution groups $U^t : \mathcal{F}_0 \to \mathcal{F}_0$ are uniquely characterized by their generator $V : D(V) \to \mathcal{F}_0$, where*

$$Vf = \lim_{t\to 0} \frac{U^t f - f}{t}.$$

   Moreover, for $\mathcal{F}_0 = L^2(\mu)$, V is skew-adjoint.

2. *The Koopman evolution group $U^t : L^\infty(\mu) \to L^\infty(\mu)$ is uniquely characterized by its generator $V : D(V) \to \mathcal{F}_0$, where*

$$Vf = \lim_{t\to 0} \frac{U^t f - f}{t}$$

   in weak-* sense.

Generator of Koopman evolution groups

Theorem 5.8 (ter Elst & Lemańczyk).

Let (Ω, Σ) be a compact metrizable space equipped with its Borel σ-algebra Σ. Let μ be a Borel probability measure on Ω and $U^t : L^2(\mu) \to L^2(\mu)$ a C_0 unitary evolution group with generator $V : D(V) \to L^2(\mu)$. Then, the following are equivalent.

1. *For every $t \in \mathbb{R}$ there exists a μ-a.e. invertible, measurable, and measure-preserving flow $\Phi^t : \Omega \to \Omega$ such that $U^t f = f \circ \Phi^t$.*
2. *The space $\mathfrak{A}(V) = D(V) \cap L^\infty(\mu)$ is an algebra with respect to function multiplication, and V is a derivation on $\mathfrak{A}$:*

$$V(fg) = (Vf)g + f(Vg), \quad \forall f, g \in \mathfrak{A}(V).$$

Counter-example. Let $V = i\Delta$, Δ Laplacian (2nd order self-adjoint operator).

Δ does not satisfy the Leibniz rule, i.e. $\exists f, g$ s.t. $\Delta(fg) = (\Delta f)g + f(\Delta g)$.

Thus, the unitary group generated by $i\Delta$, $U^t = e^{it\Delta}$, is not a Koopman group, i.e. there is no classical flow $\Phi^t : \Omega \to \Omega$ s.t. $U^t f = f \circ \Phi^t$.

Point spectrum

Spectrum: $\sigma(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ has no bounded inverse}\}$

i.e. there is no bounded operator B s.t.

$(A-\lambda I) \circ B f = f$ for all $f \in E$ s.t. $Bf \in D(A)$

$B(A-\lambda I) f = f$ for all $f \in D(A)$.

Definition 5.9.

Let $A : D(A) \to E$ be an operator on a Banach space with domain $D(A) \subseteq E$. The point spectrum of A, denoted as $\sigma_p(A) \subseteq \mathbb{C}$ is defined as the set of its eigenvalues. That is, $\lambda \in \mathbb{C}$ is an element of $\sigma_p(A)$ iff there is a nonzero vector $u \in E$ (an eigenvector) such that

Note: $\sigma_p(A) \subseteq \sigma(A)$

$$Au = \lambda u.$$

Notation.

- We use the notation $\sigma_p(A; E)$ when we wish to make explicit the Banach space on which A acts.

Eigenvalues and eigenfunctions

Definition 5.10.
Let $A : D(A) \to E$ be the generator of a C_0 semigroup $\{S^t\}_{t\geq 0}$ on a Banach space E. We say that $\lambda \in \mathbb{C}$ is an eigenvalue of the semigroup if λ is an eigenvalue of A, i.e., there exists a nonzero $u \in D(A)$ such that

$$Au = \lambda u.$$

Lemma 5.11.
With notation as above, λ is an eigenvalue of $\{S^t\}$ if and only if $\overset{u}{\cancel{z}}$ is an eigenvector of S^t for all $t \geq 0$, i.e., there exist $\Lambda^t \in \mathbb{C}$ such that

$$S^t u = \Lambda^t u, \quad \forall t \geq 0.$$

In particular, we have $\Lambda^t = e^{\lambda t}$.

Eigenvalues of the generator on $L^2(\mu)$ for measure-preserving flows:

Let $Vu = \lambda u$ with $\|u\|_{L^2(\mu)} = 1$. We have

$$\lambda = \lambda\langle u, u\rangle = \langle u, \lambda u\rangle = \langle u, Vu\rangle = \langle V^* u, u\rangle = -\langle Vu, u\rangle = -\langle \lambda u, u\rangle = -\lambda^*\langle u,u\rangle$$
$$= -\lambda^*$$

$\Rightarrow \lambda = -\lambda^* \Rightarrow \lambda = i\alpha$ for $\alpha \in \mathbb{R}$.

eigenvalues lie in the imaginary line

"eigenfrequency"

Recall from thm 5.6 that $\mathcal{A}(V) = D(V) \cap L^\infty(\mu)$ is an algebra and V acts on $\mathcal{A}(V)$ as a derivation. Given $u_1, u_2 \in \mathcal{A}(V)$ eigenvectors corresponding to eigenvalues λ_1, λ_2 then

$$V(u_1u_2) = (Vu_1)u_2 + u_1(Vu_2) = \lambda_1 u_1u_2 + \lambda_2 u_1 u_2 = (\lambda_1 + \lambda_2) u_1 u_2 \quad (1)$$

$\Rightarrow u_1u_2$ is an eigenvector corresponding to eigenvalue $\lambda_1 + \lambda_2$

Moreover since $U^t f = f \circ \phi^t$, $(U^t f)^* = (f \circ \phi^t)^* = f^* \circ \phi^t = U^t(f^*)$ and thus, $(Vf)^* = V(f^*)$ for any $f \in D(V)$.

$\Rightarrow$ If $Vu = \lambda u$ and $\lambda = -\lambda^*$, then $Vu^* = (Vu)^* = (\lambda u)^* = -\lambda u^*$

$\Rightarrow u^*$ is an eigenvector corresponding to eigenvalue $-\lambda$. $\quad (2)$

Also, we have $V\mathbb{1} = 0\mathbb{1} \Rightarrow 0 \in \sigma_p(V) \quad (3)$

$\sigma_p(V)$ has the structure of an additive group

Fact. For a measure-preserving flow, every eigenfunction u of V is a periodic observable with period $2\pi/\alpha$ where α is the corresponding eigenfrequency:

$$U^t u = \Lambda^t u = e^{\lambda t} u = e^{i\alpha t} u \Rightarrow U^{t+2\pi/\alpha} u = u$$

Point spectra for measure-preserving flows

Theorem 5.12.
Let $\Phi^t : \Omega \to \Omega$ a be a measure-preserving flow of a probability space (Ω, Σ, μ). Let $U^t : L^p(\mu) \to L^p(\mu)$ be the associated Koopman operators on $L^p(\mu)$, $p \in [1, \infty]$, and $V : D(V) \to L^p(\mu)$ the corresponding generators. Then, the following hold.

1. *For every $p, q \in [1, \infty]$ and $t \in \mathbb{R}$, $\sigma_p(U^t, L^p(\mu)) = \sigma_p(U^t, L^q(\mu))$.*
2. $\sigma_p(V, L^p(\mu)) = \sigma_p(V, L^q(\mu))$.
3. $\sigma_p(U^t)$ *is a subgroup of* S^1. $\sim \Lambda_1^t, \Lambda_2^t \in \sigma_p(U^t) \Rightarrow \Lambda_1^t \Lambda_2^t \in \sigma_p(U^t)$
4. $\sigma_p(V)$ *is a subgroup of* $i\mathbb{R}$.

Corollary 5.13.
Every eigenfunction of V lies in $L^\infty(\mu)$, and thus in $L^p(\mu)$ for every $p \in [1, \infty]$.

Given $\lambda = i\alpha \in \sigma_p(V)$, we say that α is an eigenfrequency of V.

Generating frequencies

Definition 5.14.
Assume the notation of Theorem 5.12.

1. We say that $\{ia_0, ia_1, \ldots\} \subseteq \sigma_p(V)$ is a generating set if for every $i\alpha \in \sigma_p(V)$ there exist $j_1, j_2, \ldots, j_n \in \mathbb{Z}$ and $k_1, k_2, \ldots, k_n \in \mathbb{N}$ such that
$$\alpha = j_1 \overset{a}{\cancel{\alpha}}_{k_1} + j_2 \overset{a}{\cancel{\alpha}}_{k_2} + \ldots + j_n \overset{a}{\cancel{\alpha}}_{k_n}.$$
2. We say that $\sigma_p(V)$ is finitely generated if it has a finite generating set.
3. A generating set is said to be minimal if it does does not have any proper subsets which are generating sets.

Lemma 5.15.

1. *The elements of a minimal generating set are rationally independent.*
2. *If a minimal generating set has at least two elements, then $\sigma_p(V)$ is a dense subset of the imaginary line.*

Example Ergodic rotation on $\mathbb{T}^2$

$$\phi^t(\underbrace{\omega_1, \omega_2}) = (\omega_1 + \alpha_1 t, \omega_2 + \alpha_2 t) \bmod 2\pi, \qquad \alpha_1, \alpha_2 \in \mathbb{R}, \text{ rationally independent.}$$

$\omega = (\omega_1, \omega_2) \in \mathbb{T}^2$

Vector field: $v(\omega_1, \omega_2) = (\alpha_1, \alpha_2)$, $\dot{\omega}(t) = v(\omega(t))$

Generator: $$Vf(\omega_1, \omega_2) = \lim_{t \to 0} \frac{U^t f(\omega_1, \omega_2) - f(\omega_1, \omega_2)}{t} = \alpha_1 \frac{\partial f}{\partial \omega_1} f(\omega_1, \omega_2) + \alpha_2 \frac{\partial f}{\partial \omega_2} f(\omega_1, \omega_2)$$

$= v \cdot Df$ for $f \in C^1(\mathbb{T}^2)$

Consider the Fourier functions $\phi_j(\omega_1, \omega_2) = e^{i(j_1 \omega_1 + j_2 \omega_2)}$ $\quad j = (j_1, j_2) \in \mathbb{Z}^2$.

We get $$V\phi_j = i \underbrace{(\alpha_1 j_1 + \alpha_2 j_2)}_{\text{eigenfrequency } \alpha_j} \phi_j$$

Since $\{\phi_j\}_{j \in \mathbb{Z}^2}$ forms an orthonormal basis of $L^2(\mu)$ we have computed all e-funcs of V

$$\sigma_p(V) = \{i\alpha_j\}_{j \in \mathbb{Z}^2}$$

$\{i\alpha_1, i\alpha_2\}$ is a minimal generating set

Generating frequencies

Lemma 5.16.
Let $g_1, g_2, \ldots$ be eigenfunctions corresponding to the eigenvalues of the generating set in Definition 5.14, i.e., $Vg_j = i\alpha_j g_j$. Then, for every $i\alpha \in \sigma_p(V)$ with $\alpha = j_1\alpha_{k_1} + j_2\alpha_{k_2} + \ldots + j_n\alpha_{k_n}$,

$$z = g_{k_1}^{j_1} g_{k_2}^{j_2} \cdots g_{k_n}^{j_n}$$

is an eigenfunction of V corresponding to the eigenfrequency α.

Invariant subspaces

Notation.

- $H_p = \overline{\operatorname{span}\{u \in L^2(\mu) \colon u \text{ is an eigenfunction of } V\}}$.
- $H_c = H_p^\perp$.
- $\{z_0, z_1, \dots\}$: Orthonormal eigenbasis of H_p, $Vz_j = i\alpha_j z_j$.

Theorem 5.17.
Let $\Phi^t : \Omega \to \Omega$ be a measure-preserving flow on a completely metrizable space with an invariant probability measure μ.

1. *H_p and H_c are U^t-invariant subspaces.*
2. *Every $f \in H_p$ satisfies*

$$U^t f = \sum_{j=0}^{\infty} \hat{f}_j e^{i\alpha_j t} z_j, \quad \hat{f}_j = \langle z_j, f \rangle_{L^2(\mu)}.$$

3. *Every $f \in H_c$ satisfies*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T |\langle g, U^t f \rangle_{L^2(\mu)}| = 0, \quad \forall g \in L^2(\mu).$$

Pure point spectrum

Definition 5.18.
With the notation of Theorem 5.17, we say that a measure-preserving flow $\Phi^t : \Omega \to \Omega$ has pure point spectrum if $H_p = L^2(\mu)$.

$\Leftrightarrow$ V is diagonalizable

Remark 5.19.
For a system with pure point spectrum:

1. The spectrum of V is not necessarily discrete.
2. The continuous spectrum is not necessarily empty.

↳ for pure-point spectrum systems $\sigma(V) = \overline{\sigma_p(V)}$

↳ E.g. for ergodic rotation on $\mathbb{T}^2$, $\sigma_p(V) = \{ i(j\alpha_1 + k\alpha_2),\ j,k \in \mathbb{Z},\ \alpha_1/\alpha_2 \notin \mathbb{Q} \}$

↳ $\sigma_p(V)$ is a dense subset of $i\mathbb{R}$

Point spectra for ergodic flows

Proposition 5.20.

With the notation of Theorem 5.12, assume that $\Phi^t : \Omega \to \Omega$ is ergodic.

1. *Every eigenvalue $\lambda \in \sigma_p(V)$ is simple.*
2. *Every corresponding eigenfunction $z \in L^p(\mu)$ normalized such that $\|z\|_{L^p(\mu)} = 1$ for any $p \in [1, \infty]$ satisfies $|z| = 1$ μ-a.e.*

Factor maps

Definition 5.21.
Let $T_1 : \Omega_1 \to \Omega_1$ and $T_2 : \Omega_2 \to \Omega_2$ be measure-preserving transformations of the probability spaces $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$. We say that T_2 is a factor of T_1 if there exists a T_1-invariant set $\cancel{S_1}\ M_1 \in \Sigma_1$ with $\mu_2(\cancel{S_{+}1}\ M_1) = 1$, a T_2-invariant set $\cancel{S_2}\ M_2 \in \Sigma_2$ with $\mu_2(\cancel{S_2}\ M_2) = 1$, and a measure-preserving, surjective map $\varphi : S_1 \to S_2$ such that

$$T_2 \circ \varphi = \varphi \circ T_1.$$

Such a map φ is called a factor map and satisfies the following commutative diagram:

$$\begin{tikzcd} M_1 \arrow[r, "T_1"] \arrow[d, "\varphi"'] & M_1 \arrow[d, "\varphi"] \\ M_2 \arrow[r, "T_2"] & M_2 \end{tikzcd} .$$

Metric isomorphisms

Definition 5.22.
With the notation of Definition 5.21, we say that T_1 and T_2 are measure-theoretically isomorphic or metrically isomorphic if there is a factor $\varphi : S_1 \to S_2$ with a measurable inverse.

Theorem 5.23 (von Neumann).
Let $\Phi^t : \Omega \to \Omega$ be a measure-preserving flow on a completely metrizable probability space (Ω, Σ, μ) with pure point spectrum. Then, Φ^t is metrically isomorphic to a translation on a compact abelian group $\mathcal{G}$. Explicitly, $\mathcal{G}$ can be chosen as the character group of the point spectrum $\sigma_p(V)$.

Metric isomorphisms

Corollary 5.24.
If $\sigma_p(V)$ is finitely generated, then Φ^t is metrically isomorphism to an ergodic rotation on the d-torus, where d is the number of generating frequencies of $\sigma_p(V)$. Explicitly, supposing that $\{i\alpha_1, \dots, i\alpha_d\}$ is a minimal generating set of $\sigma_p(V)$ with corresponding unit-norm eigenfunctions $z_1, \dots, z_d$ we have

$$R^t \circ \varphi = \varphi \circ \Phi^t,$$

where $R^t : \mathbb{T}^d \to \mathbb{T}^d$ is the torus rotation with frequencies $\alpha_1, \dots, \alpha_d$, and

$$\varphi(\omega) = (z_1(\omega), \dots, z_d(\omega)), \quad \mu\text{-a.e.}$$

$$\begin{tikzcd} \Omega \arrow[r, "\Phi^t"] \arrow[d, "\varphi"'] & \Omega \arrow[d, "\varphi"] \\ \mathbb{T}^d \arrow[r, "R^t"] & \mathbb{T}^d \end{tikzcd}$$

Spectral isomorphisms

Definition 5.25.
With the notation of Definition 5.22, let $U_1 : L^2(\mu_1) \to L^2(\mu_1)$ and $U_2 : L^2(\mu_2) \to L^2(\mu_2)$ be the Koopman operators associated with T_1 and T_2, respectively. We say that T_1 and T_2 are spectrally isomorphic if there exists a unitary map $\mathcal{U} : L^2(\mu_1) \to L^2(\mu_2)$ such that

$$U_2 \circ \mathcal{U} = \mathcal{U} \circ U_1.$$

Theorem 5.26 (von Neumann).
Two measure-preserving flows with pure point spectra are metrically isomorphic iff they are spectrally isomorphic.

Spectral isomorphisms

Definition 5.25.
With the notation of Definition 5.22, let $U_1 : L^2(\mu_1) \to L^2(\mu_1)$ and $U_2 : L^2(\mu_2) \to L^2(\mu_2)$ be the Koopman operators associated with T_1 and T_2, respectively. We say that T_1 and T_2 are spectrally isomorphic if there exists a unitary map $\mathcal{U} : L^2(\mu_1) \to L^2(\mu_2)$ such that

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Theorem 5.26 (von Neumann).
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