Section 6

Representation of classical dynamics in quantum circuits

Classical and quantum bits

Characters $\delta_1 : \mathbb{C}^2 \to \mathbb{C}$
$\delta_2 : \mathbb{C}^2 \to \mathbb{C}$
$\delta_1 \begin{pmatrix} a \\ b \end{pmatrix} = a, \quad \delta_2 \begin{pmatrix} a \\ b \end{pmatrix} = b$

- A (classical) bit is a pure state of the abelian algebra $\mathbb{C}^2$.
- A quantum bit, or qubit, is a pure state of the matrix algebra $B(\mathbb{C}^2) \simeq \mathbb{M}_2(\mathbb{C})$.
- Noisy classical bits and qubits are represented by mixed states of $\mathbb{C}^2$ and $\mathbb{M}_2$, respectively.

Quantum computers

A quantum computer is a finite-dimensional quantum mechanical system associated with the tensor product Hilbert space $\mathbb{B}_n \equiv \mathbb{B}^{\otimes n}$ with $\mathbb{B} = \mathbb{C}^2$.

$\dim \mathbb{B}_n = 2^n$

Notation.

- $|0\rangle$ and $|1\rangle$ are orthonormal basis vectors of $\mathbb{B}$ known as computational basis vectors.
- $|b_1 \cdots b_n\rangle \equiv |b_1\rangle \otimes \cdots \otimes |b_n\rangle$ are orthonormal basis vectors of $\mathbb{B}_n$.

$b_i \in \{0,1\}$

$|0\rangle \leftrightsquigarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$|1\rangle \leftrightsquigarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

TENSOR PRODUCT OF VECTOR SPACES

Given two vector spaces V_1, V_2 over a field $\mathbb{K}$, the tensor product space $V_1 \otimes V_2$ is a vector space consisting of linear combinations of elements of the form $v_1 \otimes v_2$ with $v_1 \in V_1$, $v_2 \in V_2$ under the following identifications:

$$(kv_1)\otimes v_2 = v_1 \otimes (kv_2) = k(v_1 \otimes v_2)$$

$$(u_1+v_1)\otimes v_2 = u_1\otimes v_2 + v_1 \otimes v_2 \quad , \quad \forall k \in \mathbb{K}, \quad u_1, v_1 \in V_1$$

$$v_1 \otimes (u_2+v_2) = v_1\otimes u_2 + v_1\otimes v_2 \qquad u_2, v_2 \in V_2$$

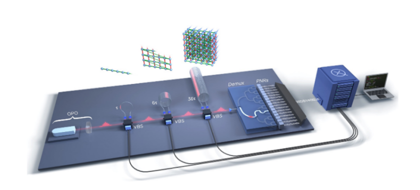
If $\dim V_1 = d_1$ and $\dim V_2 = d_2$, then $\dim V_1 \otimes V_2 = d_1 d_2$.

If V_1, V_2 are Hilbert spaces, then $V_1 \otimes V_2$ becomes a Hilbert space equipped with the inner product

$$\langle u_1 \otimes u_2 , v_1 \otimes v_2 \rangle_{V_1 \otimes V_2} = \langle u_1, v_1 \rangle_{V_1} \langle u_2, v_2 \rangle_{V_2} \quad , \quad \forall\ u_1, v_1 \in V_1, \quad u_2, v_2 \in V_2 .$$

Quantum computers

IBM Q One



AWS Borealis

Physical qubit implementations include superconducting charges, trapped ions, and photons.

Quantum circuits

A quantum circuit consists of wires, representing individual qubits, and gates representing operations (quantum channels) on qubits.

measures the runtime of the quantum algorithm implemented by the circuit.

- The depth of a quantum circuit is the longest path in the circuit.

unitary $\leftrightarrow$ quantum channel on $B(\mathcal{B}_n)$

Quantum circuits

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- The depth of a quantum circuit is the longest path in the circuit.

Goal. Given a C_0 group of unitary Koopman operators $U^t : H \to H$ induced by a measure-preserving flow with skew-adjoint generator $V : D(V) \to H$ and a subspace $H_L \subset D(V) \subset H$ of dimension 2^n, find a unitary $W : H_L \to \mathbb{B}_n$ such that e^{tG_L} with $G_L = W\Pi_L V \Pi_L W^*$ is representable by a circuit of low depth.

Generator

$$\begin{tikzcd} V \arrow[d] \\ V_L = \Pi_L V \Pi_L \arrow[d] \\ G_L = W V_L W_L^* \end{tikzcd}$$

Koopman evolution

$$\begin{tikzcd} H \arrow[r, "U^t"] \arrow[d, "\Pi_L"'] & H \arrow[d, dashed, "\Pi_L"] \\ H_L \arrow[r, "e^{tV_L}"] \arrow[d, "W"'] & H_L \arrow[d, "W"] \\ \mathbb{B}_n \arrow[r, "e^{tG_L}"] & \mathbb{B}_n \end{tikzcd}$$

\+ Efficient initialization

\+ Readout

Representation of pure states of M_2 on the Bloch sphere

Unit vector $\xi = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle \in \mathbb{C}^2 \equiv \mathbb{B}$, $\theta \in (0, 2\pi)$, $\phi \in (0, 2\pi)$

Density operator $\rho = \langle \xi, \cdot \rangle \xi \simeq \xi\xi^\dagger = \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix} \left(\cos\theta/2,\ e^{-i\phi}\sin\theta/2\right)$

$$= \begin{pmatrix} \cos^2\theta/2 & e^{-i\phi}\cos(\theta/2)\sin(\theta/2) \\ e^{i\phi}\cos\theta/2\,\sin\theta/2 & \sin^2\theta/2 \end{pmatrix} = \begin{pmatrix} (1+\cos\theta)/2 & (e^{-i\phi}\sin\theta)/2 \\ (\underbrace{e^{i\phi}}_{\cos\phi + i\sin\phi}\sin\theta)/2 & (1-\cos\theta)/2 \end{pmatrix}$$

$$= \frac{1}{2}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \cos\theta & 0 \\ 0 & -\cos\theta \end{pmatrix} + \begin{pmatrix} 0 & \sin\theta\cos\phi \\ \sin\theta\cos\phi & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i\sin\theta\sin\phi \\ i\sin\theta\sin\phi & 0 \end{pmatrix} \right)$$

$$= \frac{1}{2}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \cos\theta \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\sigma_1} + \sin\theta\cos\phi \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\sigma_2} + -\sin\theta\sin\phi \underbrace{\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}}_{\sigma_3} \right)$$

$\sigma_1 \leftrightarrow Z$, $\sigma_2 \leftrightarrow X$, $\sigma_3 \leftrightarrow Y$

Examples of quantum gates

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \leftrightsquigarrow \text{ NOT gate}$$

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$H|0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$H|1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

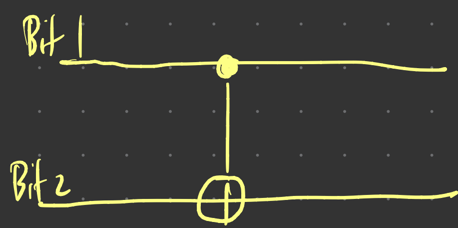
$$R_x(\theta) = e^{-\frac{1}{2} i \theta X}$$ $\leftsquigarrow$ Rotation gate by angle θ about x-axis.

$$R_y(\theta) = e^{-\frac{1}{2} i \theta Y}$$

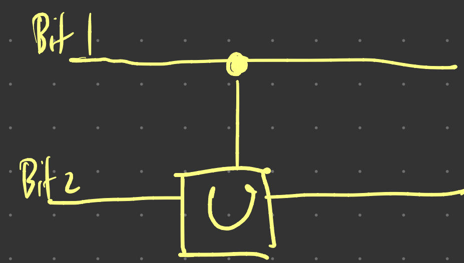
$$R_z(\theta) = e^{-\frac{1}{2} i \theta Z}$$

CNOT gate (acts on $\mathcal{B}_2 = \mathcal{B} \otimes \mathcal{B} \simeq \mathbb{C}^4$

$$\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right) \begin{pmatrix} a \\ b \\ \hline c \\ d \end{pmatrix} = \begin{pmatrix} a \\ b \\ \hline d \\ c \end{pmatrix}$$



$$\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{U}} \\ 0 & 0 & & \end{array}\right) \left(\begin{array}{c} a \\ b \\ \hline c \\ d \end{array}\right) = \left(\begin{array}{c} a \\ b \\ \hline d \\ c \end{array}\right)$$



$W : H_L \longrightarrow \mathbb{B}_n$ unitary

Given orthonormal basis $\{\phi_0, \phi_1, \dots, \phi_L\}$ of H_L, $\dim H_L = 2^n$ define

$$W \phi_\ell = |b\rangle = |b_1\rangle \otimes \dots \otimes |b_n\rangle$$

where $b = (b_1, \dots, b_n) \in \{0,1\}^n$ gives the binary representation of ℓ:

$$\ell = \sum_{i=1}^{n} b_i 2^{n-i}$$

Example: $n = 2$, $L = 4$

ℓ	b
0	0 0
1	0 1
2	1 0
3	1 1

Measurement in the quantum computational basis

Associated with the quantum computational basis $\{|b\rangle\}_{b\in\{0,1\}^n}$ of $\mathcal{B}_n$ is a projection-valued measure $\mathcal{E}_n : \Sigma_{\{0,1\}^n} \longrightarrow B(\mathcal{B}_n) \simeq M_n$

↳ σ-algebra consisting of all subsets of $\{0,1\}^n$ i.e. all binary strings of length n

$$\mathcal{E}_n(S) = \sum_{b\in S} \underbrace{|b\rangle\langle b|}$$

↳ orthogonal projection onto $|b\rangle$

That is, if the quantum computer is in a state associated with state vector $|\xi\rangle \in \mathcal{B}_n$ a measurement of $\mathcal{E}_n$ gives a binary string $b \in \{0,1\}^n$ with probability

$$\langle \xi | \mathcal{E}_n(\{b\}) | \xi \rangle$$

Rather than $\mathcal{E}_n$ ideally we would like to measure the PVM $\mathcal{F}_n$ which is the spectral measure of the observable $A_L = W(\pi_L f)W^*$

↑ projected multiplication operator

Solution: Compute eigendecomposition of A_L:

$$A_L |u_j\rangle = a_j |u_j\rangle \,, \qquad |u_j\rangle = \sum_{k=0}^{2^n-1} u_{kj} \underbrace{|b_k\rangle}_{\text{binary rep. of } k}$$

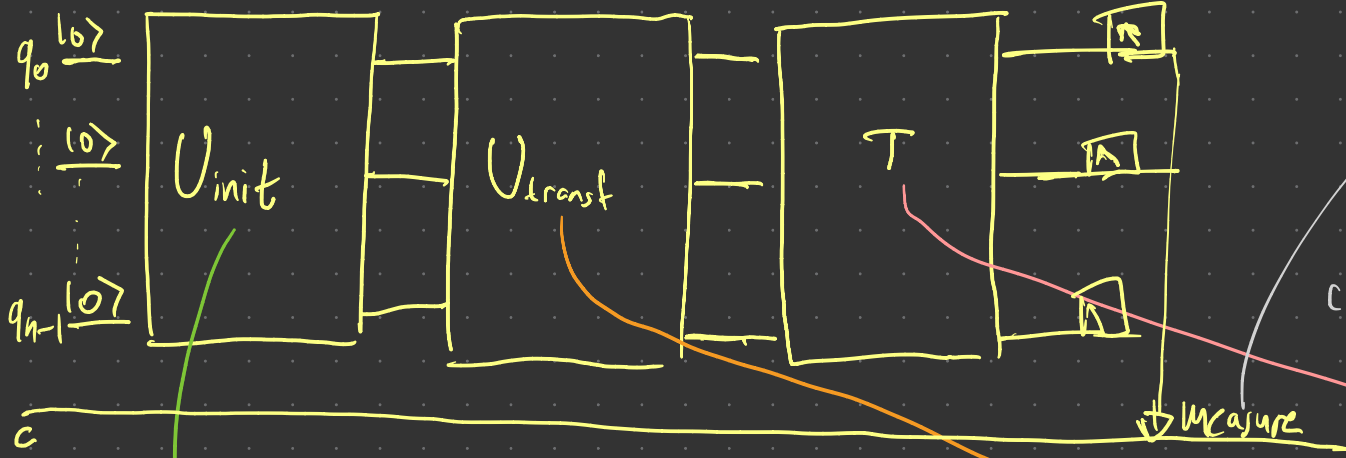
Define the unitary $T : \mathbb{B}_n \to \mathbb{B}_n$ with matrix rep.

$$\begin{pmatrix} u_{00} & \cdots & u_{0,L-1} \\ \vdots & & \vdots \\ u_{L-1,0} & & u_{L-1,L-1} \end{pmatrix}$$

Then $T^* A_L T$ is a diagonal operator in the quantum computational basis.

Then a measurement of F_n on a state $|\xi\rangle$ is equivalent to a measurement of E_n on the state vector $T|\xi\rangle$

Koopman evolution circuit



Outcome is a binary string $|b\rangle$ that represents an integer $\ell \in \{0, \dots, 2^n - 1\}$

Our prediction for the true classical evolution $Uf(x)$ is the eigenvalue a_ℓ of $\hat{A}_L$

Want: $U_{init} : \mathbb{B}_n \to \mathbb{B}_n$ unitary such that $U_{init}|0 \cdots 0\rangle = |\hat{\xi}_x\rangle$ where $|\hat{\xi}_x\rangle = W \xi_x$

$\xi_x \in H_L$ is the state vector in the corresponding to classical state x.

$U_{transf} : \mathbb{B}_n \to \mathbb{B}_n$ represents the transfer operator in the quantum computational basis, i.e.

$$U_{transf} = W U_L^* W^*$$

projected transfer operator

$$U_L^* = \Pi_L U^* \Pi_L$$

Diagonalizes the target observable $\hat{A}_L$

Finally estimate the QM expectation $\mathbb{E}_{\xi_x}(\Pi_L f)$ as a Monte Carlo average of an ensemble of such measurements $|b\rangle$.

Implementation for systems with pure point spectrum

Assume that state space dynamics $\Phi^t : \Omega \to \Omega$ is conjugate to a rotation $R^t : \mathbb{T}^d \to \mathbb{T}^d$ for some dimension d

Let $\{\phi_\ell\}_{\ell \in \mathbb{Z}^d}$ be an orthonormal basis of $H = L^2(\mu)$ (inv. meas.) such that

$$V\phi_\ell = i\alpha_\ell \phi_\ell$$

Fix $n \in \mathbb{N}$ s.t. n/d is an integer. Define $H_L = \operatorname{span}\{\phi_\ell : \ell = (\ell_1, \dots, \ell_d)$ with $\ell_i \in \{-2^{n/d-1}, \dots, -1, 1, \dots, 2^{n/d-1}\}\}$

Then $\dim H_L = 2^n$.

Define $\beta : \mathbb{Z}^d \to \{0,1\}^n$ s.t. $\beta(\ell)$ is a binary rep. of the multi-index ℓ,

$W : H_L \to \mathbb{B}_n$ as $W\phi_\ell = |\beta(\ell)\rangle$.

<u>Claim</u>: The projected generator $\tilde{V}^{*} = W \Pi_L V \Pi_L W$ is diagonal in the $\{|b\rangle\}$ basis of $\mathbb{B}_n$. Moreover,

$$\begin{aligned}\tilde{V} = {} & \tilde{h}_0 \boxed{Z} \otimes I \otimes \dots \otimes I \\ & + \tilde{h}_1\, I \otimes \boxed{Z} \otimes I \otimes \dots \otimes I \\ & \vdots \\ & + \tilde{h}_{n-1}\, I \otimes \dots \otimes I \otimes \boxed{Z}\end{aligned}$$

where the coefficients $\tilde{h}_\ell$ are given by a Walsh–Fourier transform of the function

$$b \mapsto \alpha_{\beta^{-1}(b)}$$

$$\Rightarrow e^{t\tilde{V}} = e^{t\tilde{h}_0 Z} \otimes e^{t\tilde{h}_1 Z} \otimes \dots \otimes e^{t\tilde{h}_{n-1} Z}$$