MATH 146
Current Problems in Applied Mathematics: Dynamical Systems and Quantum Information

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Dartmouth College
Winter Term 2022

Section 1

Introduction

Ergodic theory

Ergodic theory studies the statistical behavior of measurable actions of groups or semigroups on spaces.

Definition 1.1.
A left action, or flow, of a (semi)group G on a set Ω is a map $\Phi : G \times \Omega \to \Omega$ with the following properties:

1. $\Phi(e, \omega) = \omega$, for the the identity element $e \in G$ and all $\omega \in \Omega$.
2. $\Phi(gh, \omega) = \Phi(g, \Phi(h, \omega))$, for all $g, h \in G$ and $\omega \in \Omega$.

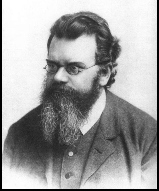
The set Ω is called the state space.

In this course, G will be an abelian group or semigroup that represents the time domain. Common choices include:

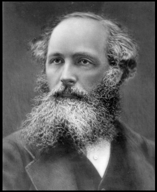
$$\mathbb{N}, \qquad \mathbb{Z}, \qquad \mathbb{R}_+, \qquad \mathbb{R}.$$

We write $\Phi^g \equiv \Phi(g, \cdot)$, $n \in \mathbb{N}, \mathbb{Z}$, and $t \in \mathbb{R}_+, \mathbb{R}$.

Ergodic theory



Ludwig Boltzmann



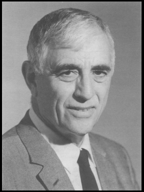
James Clerk Maxwell

Ergodic theory has its origin in the mid 19th century with the work of Boltzmann and Maxwell on statistical mechanics.

The term ergodic is an amalgamation of the Greek words ergo (έργο), which means *work*, and odos (οδός), which means *street*.

Ergodic theory


George David Birkhoff


Bernard Osgood Koopman


John von Neumann

The mathematical foundations of the subject were established by Koopman, von Neumann, Birkhoff, and many others, in work dating to the 1930s.

Modern ergodic theory is a highly diverse subject with connections to functional analysis, harmonic analysis, probability theory, topology, geometry, number theory, and other mathematical disciplines.

Observables and ergodic hypothesis

Rather than studying the flow Φ directly, ergodic theory focuses on its induced action on linear spaces of observables, e.g.,

$$\mathcal{F} = \{f : \Omega \to \mathcal{Y}\},$$

for a vector space $\mathcal{Y}$ (oftentimes, $\mathcal{Y} = \mathbb{R}$ or $\mathbb{C}$).

Drawing on intuition from mechanical systems, Boltzmann postulated that time averages of observables should well-approximate expectation values with respect to a reference distribution, μ.

This is encapsulated in the ergodic hypothesis,

$$\underbrace{\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(\Phi^n(\omega))}_{\text{time average}} = \underbrace{\int_\Omega f \, d\mu}_{\text{space average}},$$

which is stipulated to hold for typical initial conditions $\omega \in \Omega$ and observables $f : \Omega \to \mathcal{Y}$ in a suitable class.

Operator-theoretic perspective

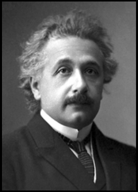
Definition 1.2.

1. For every $g \in G$, the composition operator, or Koopman operator, is the linear map $U^g : \mathcal{F} \to \mathcal{F}$ defined as
$$U^g f = f \circ \Phi^g.$$
2. The transfer operator $P^g : \mathcal{F}' \to \mathcal{F}'$ is the adjoint of U^g, defined as
$$P^g \nu = \nu \circ U^g.$$

Koopman and transfer operators allow the study of nonlinear dynamics using techniques from linear operator theory.

Quantum mechanics

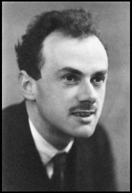

Niels Bohr


Albert Einstein


Max Planck

Quantum mechanics arose in the late 19th to early 20th century when it was realized that classical physics does not adequately describe phenomena such as the blackbody radiation spectrum, the photoelectric effect, and atomic spectral lines.

Quantum mechanics


Paul Dirac


Werner Heisenberg


Emmy Noether


Erwin Schrödinger

The mathematical formalism of quantum mechanics was developed by Schrödinger, Heisenberg, Dirac, Noether, von Neumann, and many others. The modern formulation of quantum mechanics makes heavy use of operator theory.

Dirac–von Neumann axioms of quantum mechanics

1. States are density operators, i.e., positive, trace-class operators $\rho : H \to H$ on a Hilbert space H, with $\operatorname{tr} \rho = 1$.
2. Observables are self-adjoint operators, $A : D(A) \to H$.
3. Measurement expectation and probability:

$$\mathbb{E}_\rho A = \operatorname{tr}(\rho A), \quad \mathbb{P}_\rho(\Omega) = \mathbb{E}_\rho(E(\Omega)), \quad A = \int_{\mathbb{R}} a \, dE(a).$$

4. Unitary dynamics between measurements:

$$\rho_t = U^{t*} \rho_0 U^t.$$

5. Projective measurement:

$$\rho|_e = \frac{\sqrt{e}\rho\sqrt{e}}{\operatorname{tr}(\sqrt{e}\rho\sqrt{e})}, \quad 0 < e \le I.$$

Interpretation of quantum mechanics

A vast subject in its own right, the interpretation of quantum mechanics can be approached by asking the following question:

- *Does quantum mechanics describe the world, or an observer's knowledge of the world?*

Interpretation of quantum mechanics

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- *Does quantum mechanics describe the world, or an observer's knowledge of the world?*

Quantum informational interpretations take the latter point of view.

- *Quantum information is the study of the information processing tasks that can be accomplished using quantum mechanical systems.*

Connections with ergodic theory

$$\begin{array}{lc} \text{Classical statistical} & \begin{tikzcd} P(\mu) \arrow[r, "P^t"] \arrow[d, "\Gamma"'] & P(\mu) \arrow[d, "\Gamma"] \\ \text{Quantum mechanical} & Q(H) \arrow[r, "\mathcal{P}^t"] & Q(H) \end{tikzcd} \end{array}$$

For a measure-preserving flow $\Phi^t : \Omega \to \Omega$ on a probability space (Ω, Σ, μ):

- The Koopman operator $U^t : H \to H$ is unitary on $H = L^2(\mu)$ and thus defines a quantum system.
- The classical statistical dynamics on the space of probability densities with respect to μ, $P(\mu)$, induced by the transfer operator P^t consistently embeds into the quantum dynamics induced by U^t on the space of density operators $Q(H)$.

Further reading

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[3] A. S. Holevo, *Statistical Structure of Quantum Theory* (Lecture Notes in Physics Monographs). Berlin: Springer, 2001, vol. 67.

[4] B. O. Koopman, "Hamiltonian systems and transformation in Hilbert space," *Proc. Natl. Acad. Sci.*, vol. 17, no. 5, pp. 315–318, 1931. DOI: 10.1073/pnas.17.5.315.

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Section 2

Measure-preserving transformations; Ergodic theorems

Measure-preserving dynamical systems

Definition 2.1.
Let (Ω, Σ, μ) be a measure space.

1. A measurable map $T : \Omega \to \Omega$ is said to be measure-preserving if $T_*\mu = \mu$, i.e.,
$$\mu(T^{-1}(S)) = \mu(S), \quad \forall S \in \Sigma.$$
Conversely, we say that μ is an invariant measure for T.
2. A measure-preserving map $T : \Omega \to \Omega$ is said to be invertible measure-preserving if T is bijective and T^{-1} is also measure-preserving.
3. A measurable action $\Phi : G \times \Omega \to \Omega$ is μ-preserving if $\Phi^g : \Omega \to \Omega$ is μ-preserving for every $g \in G$.

Recurrence

Theorem 2.2 (Poincaré).

Let $T : \Omega \to \Omega$ be a measure-preserving transformation of the probability space (Ω, Σ, μ). Let $S \in \Sigma$ be a measurable set with $\mu(S) > 0$. Then, under iteration by T, almost every point of S returns to S infinitely often. That is, for μ-a.e. $\omega \in S$, there exists a sequence $n_1 < n_2 < n_3 < \cdots$ of natural numbers, increasing to infinity, such that $T^{n_j}(\omega) \in S$ for all j.

Ergodicity

Definition 2.3.
Let (Ω, Σ, μ) be a probability space.

1. A measurable map $T : \Omega \to \Omega$ is said to be ergodic if for every T-invariant set, i.e., every $S \in \Sigma$ such that $T^{-1}(S) = S$ we have either $\mu(S) = 0$ or $\mu(S) = 1$.
2. A measurable action $\Phi : G \times \Omega \to \Omega$ is ergodic if for every $S \in \Sigma$ such that $\Phi^{-g}(S) = S$ for all $g \in G$ we have either $\mu(S) = 0$ or $\mu(S) = 1$.

Measure-theoretic characterization of ergodicity

Theorem 2.4.
Let $T : \Omega \to \Omega$ be a measure-preserving transformation of the probability space (Ω, Σ, μ). Then, the following are equivalent.

1. *T is ergodic.*
2. *The only measurable sets $S \in \Sigma$ such that $\mu(T^{-1}(S) \triangle S) = 0$ have either $\mu(S) = 0$ or $\mu(S) = 1$.*
3. *For every $S \in \Sigma$ with $\mu(S) > 0$, we have $\mu(\bigcup_{n=1}^{\infty} T^{-n}(S)) = 1$.*
4. *For every $R, S \in \Sigma$ with $\mu(R) > 0$ and $\mu(S) > 0$, there exists $n > 0$ with $\mu(T^{-n}(R) \cap S) > 0$.*

Measure-theoretic characterization of ergodicity

Theorem 2.5.
Let (Ω, Σ, μ) be a probability space.

1. *A measure-preserving action $\Phi : \mathbb{N} \times \Omega \to \Omega$ is ergodic iff*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu(\Phi^{-n}(R) \cap S) = \mu(R)\mu(S), \quad \forall R, S \in \Sigma.$$

2. *A measure-preserving action $\Phi : \mathbb{R}_+ \times \Omega \to \Omega$ is ergodic iff*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mu(\Phi^{-t}(R) \cap S)\, dt = \mu(R)\mu(S), \quad \forall R, S \in \Sigma.$$

Koopman operators on L^p spaces

Definition 2.6.
A measurable map $T : \Omega \to \Omega$ on a measure space (Ω, Σ, μ) is said to be nonsingular if it preserves null sets, i.e., if whenever $\mu(S) = 0$ we have $T_*\mu(S) = \mu(T^{-1}(S)) = 0$.

Notation.

- $\mathbb{L}(\Sigma) = \{f : \Omega \to \mathbb{R} : f \text{ is } \Sigma\text{-measurable}\}$.
- $L(\mu) = \{[f]_\mu : f \in \mathbb{L}(\Sigma)\}$.
- $L^p(\mu) = \{[f]_\mu \in L(\mu) : \int_\Omega |f|^p \, d\mu < \infty\}$.
- $L^\infty(\mu) = \{[f]_\mu \in L(\mu) : \operatorname{esssup}_\mu |f| < \infty\}$.

Koopman operators on L^p spaces

Proposition 2.7.

With notation as above, the following hold.

1. *If T is measurable, then the composition map $U : f \mapsto f \circ T$ maps $\mathbb{L}(\Sigma)$ into itself.*
2. *If T is nonsingular, then $\mathcal{U} : L(\mu) \to L(\mu)$ with $\mathcal{U}[f]_\mu = [Uf]_\mu$ is a well-defined linear map.*
3. *If T is nonsingular, then $L^\infty(\mu)$ is invariant under $\mathcal{U}$, i.e.,*

$$\mathcal{U} L^\infty(\mu) \subseteq L^\infty(\mu).$$

4. *If T is measure-preserving, then $\mathcal{U}$ is an isometry of $L^p(\mu)$, $1 \leq p \leq \infty$, i.e.,*

$$\|\mathcal{U}[f]_\mu\|_{L^p(\mu)} = \|[f]_\mu\|_{L^p(\mu)}.$$

5. *If T is invertible measure-preserving, then $\mathcal{U}$ is an isomorphism of $L^p(\mu)$, $1 \leq p \leq \infty$, i.e., $\mathcal{U}$ and $\mathcal{U}^{-1}$ are both isometries.*

Henceforth, we abbreviate $[f]_\mu \equiv f$, $U \equiv \mathcal{U}$.

Koopman operators on L^2

Notation.

- $\langle f_1, f_2 \rangle_{L^2(\mu)} = \int_\Omega f_1^* f_2 \, d\mu$.

The Koopman operator induced by a μ-preserving map $T : \Omega \to \Omega$ preservers Hilbert space inner products,

$$\langle U f_1, U f_2 \rangle_{L^2(\mu)} = \langle f_1, f_2 \rangle_{L^2(\mu)}.$$

If, in addition, T is invertible measure-preserving, then U is a unitary operator,

$$U^* = U^{-1}.$$

Duality of L^p spaces

Notation.
For a probability space (Ω, Σ, μ), we let:

- $M_q(\Omega, \mu) = \left\{ \text{measures } \nu \ll \mu \text{ with density } \frac{d\nu}{d\mu} \in L^q(\mu) \right\}$.
- Duality pairing: $\langle \cdot, \cdot \rangle_\mu : L^p(\mu)^* \times L^p(\mu) \to \mathbb{R}$, $\langle \alpha, f \rangle_\mu = \alpha f$.

For $1 \leq p < \infty$, we can identify functionals in $L^p(\mu)^*$ with measures in $M_q(\Omega, \mu)$, $\frac{1}{p} + \frac{1}{q} = 1$, through the map $\iota_q : M_q(\Omega, \mu) \to L^p(\mu)^*$,

$$(\iota_q \nu) f = \int_\Omega f \rho \, d\mu, \quad \rho = \frac{d\nu}{d\mu}.$$

Equipping $M_q(\Omega, \mu)$ with the norm

$$\|\nu\|_{M_q(\Omega, \nu)} = \left\| \frac{d\nu}{d\mu} \right\|_{L^q(\mu)},$$

ι_q becomes an isomorphism of Banach spaces. Thus, we have

$$L^p(\mu)^* \simeq M_q(\Omega, \mu) \simeq L^q(\mu), \quad 1 \leq p < \infty, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Transfer operators on L^p

Definition 2.8.
With the notation of Proposition 2.7, the transfer operator $P : L^1(\mu) \to L^1(\mu)$ is is the unique operator satisfying

$$\int_S Pf \, d\mu = \int_{T^{-1}(S)} f \, d\mu, \quad \forall f \in L^1(\mu).$$

We define $P : L^p(\mu) \to L^p(\mu)$, $1 < p \leq \infty$ by restriction of $P : L^1(\mu) \to L^1(\mu)$.

Proposition 2.9.
Under the identification $L^1(\mu)^* \simeq L^\infty(\mu)$, the transpose $P' : L^1(\mu)^* \to L^1(\mu)^*$ of the transfer operator $P : L^1(\mu) \to L^1(\mu)$ is identified with the Koopman operator $U : L^\infty(\mu) \to L^\infty(\mu)$; that is,

$$\int_\Omega f(Pg) \, d\mu = \int_\Omega (Uf) g \, d\mu, \quad \forall f \in L^\infty(\mu), \quad \forall g \in L^1(\mu).$$

Duality between Koopman and transfer operators

Proposition 2.10.

Let $1 \leq p < \infty$. Then, under the identification $L^p(\mu)^* \simeq L^q(\mu)$, $\frac{1}{p} + \frac{1}{q} = 1$, the following hold:

1. *The transpose $U' : L^p(\mu)^* \to L^p(\mu)^*$ of the Koopman operator $U : L^p(\mu) \to L^p(\mu)$ is identified with the transfer operator $P : L^q(\mu) \to L^q(\mu)$; that is,*

$$\langle f, Ug \rangle_\mu = \langle Pf, g \rangle_\mu, \quad \forall f \in L^q(\mu), \quad \forall g \in L^p(\mu).$$

2. *The transpose $P' : L^p(\mu)^* \to L^p(\mu)^*$ of the transfer operator $P : L^p(\mu) \to L^p(\mu)$ is identified with the Koopman operator $U : L^q(\mu) \to L^q(\mu)$; that is,*

$$\langle f, Pg \rangle_\mu = \langle Uf, g \rangle_\mu, \quad \forall f \in L^q(\mu), \quad \forall g \in L^p(\mu).$$

Duality between Koopman and transfer operators

Corollary 2.11.

1. *For* $1 < p < \infty$, $U : L^p(\mu) \to L^p(\mu)$ *and* $P : L^p(\mu) \to L^p(\mu)$ *satisfy*
$$U = U'', \quad P = P''.$$
2. *In the Hilbert space case,* $p = 2$, *we have* $P = U^*$.
3. *For* $1 \leq p \leq \infty$, P *has unit operator norm,* $\|P\|_{L^p(\mu)} = 1$.

Lemma 2.12.
With the notation of Proposition 2.8, if $T : \Omega \to \Omega$ *is invertible measure-preserving then* $P : L^p(\mu) \to L^p(\mu)$ *is the inverse of* $U : L^p(\mu) \to L^p(\mu)$, $P = U^{-1}$.

Spectral characterization of ergodicity

Observe that the Koopman operator $U : \mathcal{F} \to \mathcal{F}$ on any function space $\mathcal{F}$ has an eigenvalue equal to 1 with a constant corresponding eigenfunction, $\mathbb{1} : \Omega \to \mathbb{R}$,

$$U\mathbb{1} = \mathbb{1}, \quad \mathbb{1}(\omega) = 1.$$

Theorem 2.13.
Let $T : \Omega \to \Omega$ be a measure-preserving transformation of a probability space (Ω, Σ, μ). Then, μ is ergodic iff the eigenvalue equal to 1 of the associated Koopman operator U on $L(\mu)$ (and thus on any of the $L^p(\mu)$ spaces with $1 \leq p \leq \infty$) is simple, i.e.,

$$Uf = f \implies f = \text{const. } \mu\text{-a.e.}$$

Spectral characterization of ergodicity

Theorem 2.14.

1. *Let* $\Phi : \mathbb{N} \times \Omega \to \Omega$ *be a measure-preserving action and* U^n, $n \in \mathbb{N}$, *the associated Koopman operators on any of* $L(\mu)$ *or* $L^p(\mu)$, $1 \leq p \leq \infty$. *Then* Φ *is ergodic iff* $U^n f = f$ *for all* $n \in \mathbb{N}$ *implies that* f *is constant* μ*-a.e.*
2. *Let* $\Phi : \mathbb{R}_+ \times \Omega \to \Omega$ *be a measure-preserving action and* U^t, $t \in \mathbb{R}_+$, *the associated Koopman operators on any of* $L(\mu)$ *or* $L^p(\mu)$, $1 \leq p \leq \infty$. *Then,* Φ *is ergodic iff* $U^t f = f$ *for all* $t \in \mathbb{R}_+$ *implies that* f *is constant* μ*-a.e.*

Pointwise ergodic theorem

Theorem 2.15 (Birkhoff).

Let $T : \Omega \to \Omega$ be a measure-preserving transformation of a probability space (Ω, Σ, μ) with associated Koopman operator $U : L^1(\mu) \to L^1(\mu)$. Then, for every $f \in L^1(\mu)$ and μ-a.e. $\omega \in \Omega$,

$$f_N(\omega) := \frac{1}{N} \sum_{n=0}^{N-1} f(T^n(\omega))$$

converges to a function $\bar{f} \in L^1(\mu)$ that satisfies

$$U\bar{f} = \bar{f}, \quad \int_\Omega f \, d\mu = \int_\Omega \bar{f} \, d\mu.$$

In particular, if T is ergodic, then for μ-a.e. $\omega \in \Omega$,

$$\bar{f}(\omega) = \int_\Omega f \, d\mu.$$

Mean ergodic theorem

Theorem 2.16 (von Neumann).

Let $T : \Omega \to \Omega$ be a measure-preserving transformation of a probability space (Ω, Σ, μ) with associated Koopman operator $U : L^2(\mu) \to L^2(\mu)$. Let $\Pi : L^2(\mu) \to L^2(\mu)$ be the orthogonal projection onto the eigenspace of U corresponding to eigenvalue 1. Then, the sequence of operators $U_N = N^{-1} \sum_{n=0}^{N-1} U^n$ converges strongly to Π, i.e.,

$$\lim_{N\to\infty} U_N f = \Pi f, \quad \forall f \in L^2(\mu).$$

In particular, if T is ergodic, Π is the projection onto the 1-dimensional subspace of $L^2(\mu)$ containing μ-a.e. constant functions, i.e.,

$$\Pi f = \langle \mathbb{1}, f \rangle_{L^2(\mu)} \mathbb{1} = \left(\int_\Omega f \, d\mu \right) \mathbb{1}.$$

Topological dynamics

Of particular interest is the case where (G, τ_G) and (Ω, τ_Ω) are topological spaces and $\Phi : G \times \Omega \to \Omega$ is a continuous, and thus Borel-measurable, action. We let $\mathfrak{B}(\Omega)$ denote the Borel σ-algebra of Ω.

Definition 2.17.
The support of a measure $\mu : \mathfrak{B}(\Omega) \to [0, \infty]$ is the set

$$\operatorname{supp} \mu := \{\omega \in \Omega : \mu(N_\omega) > 0, \, \forall N_\omega \in \tau_\Omega\}.$$

Lemma 2.18.
With notation as above, the following hold.

1. $\operatorname{supp} \mu$ *is a closed (and thus Borel-measurable) subset of* Ω*.*
2. *If* Ω *is Hausdorff, and* μ *is a Radon measure, every Borel-measurable set* $S \subset \Omega \setminus \operatorname{supp} \mu$ *has* $\mu(S) = 0$*.*
3. *If* μ *is invariant under a continuous map* $T : \Omega \to \Omega$*, then* $\operatorname{supp} \mu$ *is also invariant,*

$$T^{-1}(\operatorname{supp} \mu) \subseteq \operatorname{supp} \mu.$$

Existence of invariant measures

Theorem 2.19 (Krylov-Bogoliubov).
Let (Ω, τ_Ω) *be a compact metrizable space and* $T : \Omega \to \Omega$ *a continuous map. Then, there exists an invariant Borel probability measure under* T*.*

Existence of dense orbits

Theorem 2.20.
Let (Ω, τ_Ω) *be a compact metrizable space,* $T : \Omega \to \Omega$ *a continuous map, and* μ *an ergodic, invariant Borel probability measure with* $\operatorname{supp} \mu = \Omega$*. Then,* μ*-a.e.* $\omega \in \Omega$ *has a dense orbit* $\{T^n(\omega)\}_{n=0}^{\infty}$*.*

Geometry of invariant measures

Theorem 2.21.
Let $T : \Omega \to \Omega$ be a continuous map on a compact metrizable space. Let $\mathcal{M}(\Omega; T)$ denote the set of T-invariant Borel probability measures on Ω. Then, the following hold:

1. *$\mathcal{M}(\Omega; T)$ is a weak-* compact, convex space.*
2. *μ is an extreme point of $\mathcal{M}(\Omega; T)$ iff it is ergodic.*
3. *If μ and ν are distinct, ergodic measures in $\mathcal{M}(\Omega; T)$, then they are mutually singular.*

Equidistributed sequences

Definition 2.22.
Let $T : \Omega \to \Omega$ be a continuous map on a compact metrizable space (Ω, τ_Ω) and μ a Borel probability measure. A sequence $\omega_0, \omega_1, \ldots$ with $\omega_n = T^n(\omega_0)$ is said to be μ-equidistributed if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(\omega_n) = \int_\Omega f \, d\mu, \quad \forall f \in C(\Omega).$$

Remark.
μ-equidistribution of $\omega_0, \omega_1, \ldots$ is equivalent to weak-* convergence of the sampling measures $\mu_N = N^{-1} \sum_{n=0}^{N-1} \delta_{\omega_n}$ to the measure μ.

Basin of a measure

Definition 2.23.
With the notation of Definition 2.22 the basin of μ is the set

$$\mathcal{B}(\mu) = \{\omega_0 \in \Omega : \omega_0, \omega_1, \ldots \text{ is } \mu\text{-equidistributed}\}.$$

By the pointwise ergodic theorem (Theorem 2.15), if Ω is a metrizable space and μ is an ergodic invariant measure with compact support, then μ-a.e. $\omega \in \Omega$ lies in $\mathcal{B}(\mu)$.

Observable measures

Definition 2.24.
With the notation of Definition 2.23, let ν be a reference Borel probability measure on Ω. The measure μ is said to be ν-observable if there exists a Borel set $S \in \mathfrak{B}(\Omega)$ with $\nu(S) > 0$ such that ν-a.e. $\omega \in S$ lies in $\mathcal{B}(\mu)$.

Intuitively, we think of ν as the measure from which we draw initial conditions. ν-observability of μ then means that the statistics of observables with respect to μ can be approximated from experimentally accessible initial conditions.

Koopman operators on spaces of continuous functions

Proposition 2.25.

Let $T : \Omega \to \Omega$ be a continuous map on a locally compact Hausdorff space. Then, the Koopman operator $U : f \mapsto f \circ T$ is well-defined as a linear map from $C(\Omega)$ into itself. Moreover:

1. *U is a contraction, i.e.,*

$$\|Uf\|_{C(\Omega)} \leq \|f\|_{C(\Omega)}, \quad \forall f \in C(\Omega),$$

   with equality if T is invertible.

2. *U has operator norm $\|U\| = 1$.*

3. *U has the properties*

$$U(fg) = (Uf)(Ug), \quad U(f^*) = (Uf)^*, \quad \forall f, g \in C(\Omega),$$

   i.e., it is a *-homomorphism of the C^*-algebra $C(\Omega)$.

Transfer operators on Borel measures

Notation.

- $M(\Omega)$: Space of signed Borel measures on topological space (Ω, τ_Ω).

Definition 2.26.
Let $T : \Omega \to \Omega$ be a continuous map on a compact metrizable space. The transfer operator $P : C(\Omega)^* \to C(\Omega)^*$ is the transpose (dual) operator to the Koopman operator $U : C(\Omega) \to C(\Omega)$,

$$P\alpha = \alpha \circ U.$$

Unique ergodicity

Definition 2.27.
Let $T : \Omega \to \Omega$ be a continuous map on a compact metrizable space (Ω, τ_Ω). T is said to be uniquely ergodic if there is only one T-invariant Borel probability measure.

Theorem 2.28.
With notation as above, the following are equivalent.

1. *T is uniquely ergodic.*
2. *For every $f \in C(\Omega)$, $N^{-1} \sum_{n=0}^{N-1} f(T^n(\omega))$ converges to a constant, uniformly with respect to $\omega \in \Omega$.*
3. *For every $f \in C(\Omega)$, $N^{-1} \sum_{n=0}^{N-1} f(T^n(\omega))$ converges pointwise to a constant.*
4. *There exists an invariant Borel probability measure μ such that*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n(\omega)) = \int_\Omega f \, d\mu, \quad \forall \omega \in \Omega.$$

Strong and weak continuity of continuous-time (semi)flows

Theorem 2.29.
Let $\Phi^t : \Omega \to \Omega$, $t \geq 0$, be a continuous-time, continuous, semiflow on a compact metrizable space Ω with associated Koopman operators $U^t : C(\Omega) \to C(\Omega)$. Then, as $t \to 0$, U^t converges strongly to the identity,

$$\lim_{t \to 0} \| U^t f - f \|_{C(\Omega)} = 0, \quad \forall f \in C(\Omega).$$

Theorem 2.30.
Let $\Phi^t : \Omega \to \Omega$, $t \geq 0$, be a continuous-time, measurable semiflow with invariant probability measure μ and associated Koopman operators $U^t : L^p(\mu) \to L^p(\mu)$. Then, the following hold as $t \to 0$:

1. *For $1 \leq p < \infty$, U^t converges strongly to the identity,*

$$\lim_{t \to 0} \| U^t f - f \|_{L^p(\mu)} = 0, \quad \forall f \in L^p(\mu).$$

2. *For $p = \infty$, U^t converges in weak-* sense to the identity,*

$$\lim_{t \to 0} \int_\Omega g(U^t f) \, d\mu = \int_\Omega gf \, d\mu, \quad \forall f \in L^\infty(\mu), \quad \forall g \in L^1(\mu).$$

Mixing

Recall from Theorem 2.4 that a measure-preserving transformation is ergodic iff

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu(T^{-n}(R) \cap S) = \mu(R)\mu(S), \quad \forall R, S \in \Sigma.$$

Definition 2.31.
Let $T : \Omega \to \Omega$ be a measure-preserving transformation of the probability space (Ω, Σ, μ).

1. T is said to be weak-mixing if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |\mu(T^{-n}(R) \cap S) - \mu(R)\mu(S)| = 0, \quad \forall R, S \in \Sigma.$$

2. T is said to be strong-mixing, or mixing, if

$$\lim_{n\to\infty} \mu(T^{-n}(R) \cap S) = \mu(R)\mu(S), \quad \forall R, S \in \Sigma.$$

Mixing

Theorem 2.32.
Let $T : \Omega \to \Omega$ be a measure-preserving transformation of the probability space (Ω, Σ, μ). Then, the following are equivalent.

1. *T is weak-mixing.*
2. *There is a subset $\mathcal{N} \subset \mathbb{N}$ of zero density such that*

$$\lim_{\substack{n\to\infty \\ n\notin\mathcal{N}}} \mu(T^{-n}(R) \cap S) = \mu(R)\mu(S), \quad \forall R, S \in \Sigma.$$

Observable-centric characterization of ergodicity and mixing

Let $T : \Omega \to \Omega$ be a measure-preserving transformation of the probability space (Ω, Σ, μ). Let $U : L^2(\mu) \to L^2(\mu)$ be the associated Koopman operator on L^2.

For $f, g \in L^2(\mu)$, define the cross-correlation function $C_{fg} : \mathbb{N} \to \mathbb{R}$, where

$$C_{fg}(n) = \langle f, U^n g \rangle_{L^2(\mu)},$$

and the autocorrelation function $C_f = C_{ff}$.

Consider also the expectation values $\bar{f} = \int_\Omega f \, d\mu$ and $\bar{g} = \int_\Omega g \, d\mu$.

Theorem 2.33.
With notation as above, the following are equivalent.

1. *T is ergodic.*
2. *For all $f, g \in L^2(\mu)$, $\lim_{n\to\infty} N^{-1} \sum_{n=0}^{N-1} C_{fg}(n) = \bar{f}\bar{g}$.*
3. *For all $f \in L^2(\mu)$, $\lim_{n\to\infty} N^{-1} \sum_{n=0}^{N-1} C_f(n) = \bar{f}^2$.*

Observable-centric characterization of ergodicity and mixing

Theorem 2.34.
With notation as above, the following are equivalent.

1. T *is weak-mixing.*
2. *For all* $f, g \in L^2(\mu)$, $\lim_{N\to\infty} N^{-1} \sum_{n=0}^{N-1} |C_{fg}(n) - \bar{f}\bar{g}| = 0$.
3. *For all* $f \in L^2(\mu)$, $\lim_{N\to\infty} N^{-1} \sum_{n=0}^{N-1} |C_f(n) - \bar{f}^2| = 0$.

Theorem 2.35.
With notation as above, the following are equivalent.

1. T *is mixing.*
2. *For all* $f, g \in L^2(\mu)$, $\lim_{N\to\infty} C_{fg}(n) = \bar{f}\bar{g}$.
3. *For all* $f \in L^2(\mu)$, $\lim_{N\to\infty} C_f(n) = \bar{f}^2$.

Spectral characterization of mixing

Theorem 2.36.
Let $T : \Omega \to \Omega$ *be a measure-preserving transformation of the probability space* (Ω, Σ, μ)*, and* $U : L^2(\mu) \to L^2(\mu)$ *the corresponding Koopman operator. Then,* T *is weak-mixing iff the only eigenvalue of* U *is the eigenvalue equal to 1.*

Mixing and product flows

Theorem 2.37.
Let $T : \Omega \to \Omega$ be a measure-preserving transformation of the probability space (Ω, Σ, μ). Then, the following are equivalent.

1. *T is weak-mixing.*
2. *$T \times T$ is ergodic with respect to the product measure $\mu \times \mu$.*
3. *$T \times T$ is weak-mixing with respect to the product measure $\mu \times \mu$.*

Further reading

[1] V. Baladi, *Positive Transfer Operators and Decay of Correlations* (Advanced Series in Nonlinear Dynamics). Singapore: World Scientific, 2000, vol. 16.

[2] N. Edeko, M. Gerlach, and V. Kühner, "Measure-preserving semiflows and one-parameter Koopman semigrpoups," *Semigr. Forum*, vol. 98, pp. 48–63, 2019. DOI: 10.1007/s00233-018-9960-3.

[3] T. Eisner, B. Farkas, M. Haase, and R. Nagel, *Operator Theoretic Aspects of Ergodic Theory* (Graduate Texts in Mathematics). Cham: Springer, 2015, vol. 272.

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Section 3

Introduction to operator algebras

Algebras – basic definitions

Definition 3.1.
An algebra (over the complex numbers) is a $\mathbb{C}$-vector space $\mathcal{A}$, equipped with a binary operation $\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ such that for every $a, b, c \in \mathcal{A}$ and $\lambda \in \mathbb{C}$, we have:

- $(ab)c = a(bc)$.
- $a(b + c) = ab + ac$.
- $(a + b)c = ac + bc$.
- $(\lambda a)b = \lambda(ab) = a(\lambda b)$.

Algebras – basic definitions

Definition 3.2.
An algebra $\mathcal{A}$ is said to be:

1. Abelian if $ab = ba$ for all $a, b \in \mathcal{A}$.
2. Unital if there is a (unique) nonzero element $\mathbb{1} \in \mathcal{A}$ such that $\mathbb{1}a = a\mathbb{1} = a$ for all $a \in \mathcal{A}$.

*-algebras

Definition 3.3.
A *-algebra (or involutive algebra) is an algebra $\mathcal{A}$ equipped with an operation $^* : \mathcal{A} \to \mathcal{A}$ such that for all $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{C}$,

- $(a^*)^* = a$.
- $(a+b)^* = a^* + b^*$.
- $(ab)^* = b^* a^*$.
- $(\lambda a)^* = \lambda^* a^*$.

Banach algebras; C^*-algebras

Definition 3.4.

1. A normed algebra is an algebra $\mathcal{A}$ equipped with a norm $\|\cdot\|$ such that
$$\|ab\| \leq \|a\|\|b\|, \quad \forall a, b \in \mathcal{A}.$$
2. A Banach algebra is a normed algebra $(\mathcal{A}, \|\cdot\|)$ which is complete with respect to $\|\cdot\|$.
3. A Banach *-algebra is a Banach algebra which is also a *-algebra.
4. A C^*-algebra $\mathcal{A}$ is a Banach *-algebra such that
$$\|a^* a\| = \|a\|^2, \quad \forall a \in \mathcal{A}.$$

- For a unital normed algebra, we can choose the norm such that $\|\mathbb{1}\| = 1$ without loss of generality.
- Henceforth, we will consider that all Banach *-algebras have isometric involution, i.e., $\|a^*\| = |a\|$.

Banach algebras; C^*-algebras

Definition 3.5.

1. Given an algebra $\mathcal{A}$, then for a subset $S \subseteq \mathcal{A}$ we denote by $\mathrm{alg}(S)$ the subalgebra of $\mathcal{A}$ generated by S, which consists of all linear combinations of finite products of elements of S. Equivalently, $\mathrm{alg}(S)$ is the smallest subalgebra of $\mathcal{A}$ containing S.
2. If $\mathcal{A}$ is a Banach algebra, the closure $\overline{\mathrm{alg}(S)}$ is said to be the Banach subalgebra of $\mathcal{A}$ generated by S.

Inverse

Definition 3.6.
An element a of a unital algebra $\mathcal{A}$ is said to be invertible if there exists a (unique) element $b \in \mathcal{A}$ such that $ab = ba = \mathbb{1}$. We write $b = a^{-1}$ and call a^{-1} the inverse of a.

We denote the set of invertible elements of $\mathcal{A}$ as $G(\mathcal{A})$. This set forms a multiplicative subgroup of $\mathcal{A}$.

Proposition 3.7.
For a unital Banach algebra $\mathcal{A}$, $G(\mathcal{A})$ is an open set and $^{-1} : G(\mathcal{A}) \to \mathcal{A}$ is continuous.

Normal elements

Definition 3.8.
An element a of a *-algebra is said to be:

1. Normal if it commutes with a^*, i.e., $aa^* - a^*a = 0$.
2. Self-adjoint if $a^* = a$.
3. Skew-adjoint if $a^* = -a$.

Given a unital Banach algebra $\mathcal{A}$ and an element $a \in \mathcal{A}$ we denote the Banach algebra generated by $\{\mathbb{1}, a\}$ as $B(a)$. If, in addition, $\mathcal{A}$ is a *-algebra, we let $B^*(a)$ be the Banach *-algebra generated by $\{\mathbb{1}, a, a^*\}$.

Lemma 3.9.
If $a \in \mathcal{A}$ is a normal element of a Banach *-algebra, then $B^*(a)$ is abelian.

Spectrum

Definition 3.10.
For an element $a \in \mathcal{A}$ of a unital Banach algebra $\mathcal{A}$ we define:

1. The spectrum as the set of complex numbers
$$\sigma(a) = \{\lambda \in \mathbb{C} : a - \lambda \notin G(\mathcal{A})\}.$$
2. The spectral radius
$$r(a) = \sup_{\lambda \in \sigma(a)} |\lambda|.$$

Theorem 3.11.
With notation as above, the following hold:

1. *$\sigma(a)$ is a compact subset of $\mathbb{C}$ such that*
$$\sup_{\lambda \in \sigma(a)} |\lambda| \leq \|a\|.$$
2. $r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$.
3. *If a is a normal element of a C^*-algebra, then $r(a) = \|a\|$.*

Homomorphisms

Definition 3.12.

1. A homomorphism $\pi : \mathcal{A} \to \mathcal{B}$ between algebras is a linear map that is compatible with algebraic multiplication, i.e.,

$$\pi(aa') = \pi(a)\pi(a'), \quad \forall a, a' \in \mathcal{A}.$$

2. A homomorphism $\pi : \mathcal{A} \to \mathcal{B}$ is said to be unital if $\mathcal{A}$ and $\mathcal{B}$ are unital and $\pi(\mathbb{1}_{\mathcal{A}}) = \mathbb{1}_{\mathcal{B}}$.
3. A homomorphism $\pi : \mathcal{A} \to \mathcal{B}$ between *-algebras is said to be a *-homomorphism if

$$\pi(a^*) = (\pi a)^*, \quad \forall a \in \mathcal{A}.$$

Representations

Definition 3.13.

1. For an algebra $\mathcal{A}$, a representation is a homomorphism $\pi : \mathcal{A} \to L(V)$, where $L(V)$ is the algebra of linear maps on a vector space V.
2. If $\mathcal{A}$ is a Banach algebra, a representation is a homomorphism $\pi : \mathcal{A} \to B(E)$, where $B(E)$ is the Banach algebra of bounded linear maps on a Banach space E.
3. If $\mathcal{A}$ is a Banach *-algebra, a *-representation is a $*$-homomorphism $\pi : \mathcal{A} \to B(H)$, where $B(H)$ is the C^*-algebra of bounded linear maps on a Hilbert space H.
4. If $\ker \pi = \{0\}$, π is said to be a faithful representation.

Representations

Definition 3.14.
For a Banach algebra $\mathcal{A}$, the left regular representation (or left multiplier representation) $\pi : \mathcal{A} \to B(\mathcal{A})$ is defined as

$$(\pi a)b = ab, \quad \forall a, b \in \mathcal{A}.$$

Proposition 3.15.

1. *The left regular representation $\pi : \mathcal{A} \to L(\mathcal{A})$ of a unital algebra $\mathcal{A}$ is faithful.*
2. *If $\mathcal{A}$ is a Banach algebra, then π is a contraction; that is, $\|\pi\| \leq 1$.*
3. *If $\mathcal{A}$ is a C^*-algebra, then π is an isometry; that is, $\|\pi\| = 1$.*

Representations of C^*-algebras

Lemma 3.16.
Let H be a Hilbert space. Then, any norm-closed *-subalgebra $\mathcal{A}$ of $B(H)$ is a C^*-algebra. We refer to every such $\mathcal{A}$ as a concrete C^*-algebra.

Theorem 3.17 (Gelfand–Naimark–Segal).
Every C^*-algebra $\mathcal{A}$ admits admits a faithful representation $\pi : \mathcal{A} \to B(H)$ on some Hilbert space H.

Characters

Definition 3.18.
A character (or multiplicative linear functional) of a unital Banach algebra $\mathcal{A}$ is a nonzero homomorphism $\chi : \mathcal{A} \to \mathbb{C}$.

Lemma 3.19.
Every character $\chi : \mathcal{A} \to \mathbb{C}$ is:

1. *Unital.*
2. *Surjective.*
3. *Contractive, i.e., $\|\chi\| \leq 1$.*

Moreover, if $\mathcal{A}$ is a C^*-algebra, then:

4. *χ is a *-homomorphism.*
5. *$\|\chi\| = 1$.*

Corollary 3.20.
Every character of a unital Banach algebra is continuous.

Characters

Proposition 3.21.
An abelian unital Banach algebra has at least one character.

Ideals

Definition 3.22.
A subalgebra $\mathcal{I} \subseteq \mathcal{A}$ of an algebra is said to be a (two-sided) ideal if $a\mathcal{I} \subseteq \mathcal{I}$ and $\mathcal{I}a \subseteq \mathcal{I}$ for all $a \in \mathcal{A}$.

Definition 3.23.
A maximal ideal is a proper ideal $\mathcal{I} \subset \mathcal{A}$ that is not a subset of any other proper ideals.

Proposition 3.24.
Every maximal ideal in a unital Banach algebra is closed.

Spectra of abelian Banach algebras

Definition 3.25.
Let $\mathcal{A}$ be a unital, abelian Banach algebra. The spectrum of $\mathcal{A}$, denoted as $\sigma(\mathcal{A})$, is the set of its characters.

Theorem 3.26 (Gelfand–Mazur).
Let $\mathcal{A}$ be an abelian unital Banach algebra. There is a canonical bijection between $\sigma(\mathcal{A})$ and the set of maximal ideals of $\mathcal{A}$. Specifically, for every $\chi \in \sigma(\mathcal{A})$, $\ker \chi$ is a maximal ideal, and every maximal ideal has this form for a unique character $\chi \in \sigma(\mathcal{A})$.

Gelfand transform

Theorem 3.27.
The spectrum $\sigma(\mathcal{A})$ of an abelian unital Banach algebra is a weak-* compact subset of $\mathcal{A}^*$. Moreover, the map $\Gamma : \mathcal{A} \to C(\sigma(\mathcal{A}))$ defined as $\Gamma(a) \equiv \hat{a}$ with $\hat{a}(\chi) = \chi(a)$ is a Banach algebra homomorphism with norm $\|\Gamma\| \leq 1$.

Definition 3.28.
The map $\Gamma : \mathcal{A} \to C(\sigma(\mathcal{A}))$ is called the Gelfand transform for $\mathcal{A}$.

Proposition 3.29.
The Gelfand transform for $\mathcal{A}$ is injective iff the intersection of all the maximal ideals of $\mathcal{A}$ is $\{0\}$. In that case, we say that $\mathcal{A}$ is semisimple.

Gelfand transform

Proposition 3.30.

For an element a of an abelian, unital, Banach algebra $\mathcal{A}$ we have

$$\sigma(a) = \operatorname{ran} \hat{a}.$$

Proposition 3.31.

Let $\mathcal{A}$ be an abelian Banach algebra generated by $\{\mathbb{1}, a\}$. Then, the map $\beta : \sigma(\mathcal{A}) \to \sigma(a)$ defined as $\beta(\chi) = \hat{a}(\chi)$ is a homeomorphism between the spectrum of $\mathcal{A}$ and the spectrum of a.

Spectra of C^*-algebras

Theorem 3.32 (Gelfand).

Let $\mathcal{A}$ be a unital, abelian C^*-algebra. Then, the Gelfand transform $\Gamma : \mathcal{A} \to C(\sigma(\mathcal{A}))$ is an isometric *-isomorphism between $\mathcal{A}$ and the C^*-algebra of continuous functions on $\sigma(\mathcal{A})$.

Theorem 3.33 (Stone).

Let X be a compact Hausdorff space. For $x \in X$ let $\delta_x \in C(X)^*$ denote the evaluation functional $\delta_x f = f(x)$. Then, the following hold.

1. $\sigma(C(X)) = \{\delta_x : x \in X\}$.
2. *X is homeomorphic to $\sigma(C(X))$ under the map $x \mapsto \delta_x$.*

Corollary 3.34.

Let X and Y be compact Hausdorff spaces. Then, X and Y are homeomorphic iff $C(X)$ and $C(Y)$ are algebraically isomorphic. In that case, $C(X)$ and $C(Y)$ are isometrically *-isomorphic C^*-algebras.

Spectra of C^*-algebras

Based on Theorems 3.32 and 3.33, we can identify unital abelian C^*-algebras with spaces of continuous functions on compact Hausdorff spaces. Generalizing this interpretation, we can interpret non-abelian C^* algebras as spaces of continuous functions on "non-commutative spaces".

Continuous functional calculus

Let a be a normal element of a unital C^*-algebra $\mathcal{A}$. Given a continuous function $f : \sigma(a) \to \sigma(a)$, we define $f(a) \in \mathcal{A}$ as

$$f(a) = \Gamma^{-1}(f \circ \beta),$$

where $\Gamma : C^*(a) \to C(\sigma(C^*(a)))$ is the Gelfand transform associated with the abelian C^*-algebra generated by a, and $\beta : \sigma(C^*(a)) \to \sigma(a)$ is the homeomorphism from Proposition 3.31.

Positive elements

Definition 3.35.
An element a of a *-algebra $\mathcal{A}$ is said to be positive if $a = b^* b$ for some $b \in \mathcal{A}$.

Definition 3.36.
A *-algebra $\mathcal{A}$ is said to be:

1. Hermitian if every self-adjoint element has real spectrum, i.e., $a \in \mathcal{A}$ and $a^* = a$ implies $\sigma(a) \subset \mathbb{R}$.
2. Symmetric if every positive element has positive spectrum, i.e., $a \in \mathcal{A}$ and $a \geq 0$ implies $\sigma(a) \subset \mathbb{R}_+$.

Theorem 3.37.
A Banach *-algebra $\mathcal{A}$ is Hermitian iff it is symmetric.

Positive elements of C^*-algebras

Theorem 3.38.
Let $\mathcal{A}$ be a C^*-algebra. The following are equivalent:

1. *a is positive (i.e., $a = b^*b$ for some $b \in \mathcal{A}$).*
2. *a is normal and $\sigma(a) \subset [0, \infty)$.*
3. *There exists a self-adjoint element $b \in \mathcal{A}$ such that $a = b^2$.*

Corollary 3.39.
Every positive element $a \in \mathcal{A}$ has a unique positive square root, i.e., a positive element $b \in \mathcal{A}$ such that $a = b^2$. We write $b = \sqrt{a}$.

Notation.
For a C^*-algebra $\mathcal{A}$:

- $\mathcal{A}_{\mathrm{sa}} \subset \mathcal{A}$ is the subspace of the self-adjoint adjoint elements.
- $\mathcal{A}_+ \subset \mathcal{A}_{\mathrm{sa}}$ is the subset of positive elements.

Positive elements of C^*-algebras

Theorem 3.40.
The set of positive elements of a C^* algebra is a convex cone, i.e.,

1. *For all $a \in \mathcal{A}_+$ and $\lambda \geq 0$, $\lambda a \in \mathcal{A}_+$.*
2. *For all $a, b \in \mathcal{A}_+$ and $\lambda \in [0,1]$, $\lambda a + (1 - \lambda b) \in \mathcal{A}_+$.*

Moreover, $\mathcal{A}_+$ is closed in the norm topology of $\mathcal{A}$.

By Theorem 3.40, positivity defines an order on $\mathcal{A}_{\mathrm{sa}}$.

- If $a \in \mathcal{A}_{\mathrm{sa}}$ is positive, we write $a \geq 0$.
- Given $a, b \in \mathcal{A}_{\mathrm{sa}}$, we write $a \leq b$ if $b - a \geq 0$.

Proposition 3.41.
Given two positive elements $a, b \in \mathcal{A}_+$ with $a \leq b$ the following hold:

1. $\|a\| \leq \|b\|$.
2. $\sqrt{a} \leq \sqrt{b}$.
3. *If $\mathcal{A}$ is unital and a, b are invertible, then $b^{-1} \leq a^{-1}$.*

States

Definition 3.42.
A linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ on a *-algebra $\mathcal{A}$ is said to be positive if $\varphi a \geq 0$ whenever a is positive.

Definition 3.43.
A state $\varphi : \mathcal{A} \to \mathbb{C}$ on a unital *-algebra $\mathcal{A}$ is a positive, linear unital functional, i.e.:

- $\varphi(a^* a) \geq 0$ for all $a \in \mathcal{A}$.
- $\varphi \mathbb{1} = 1$.

The state space of $\mathcal{A}$ is the set of its states, denoted as $S(\mathcal{A})$.

States on C^*-algebras

Proposition 3.44.

The following hold for every state $\varphi \in S(\mathcal{A})$ of a unital C^*-algebra and elements $a, b \in \mathcal{A}$.

1. $\varphi(a^*) = (\varphi a)^*$, *for all* $a \in \mathcal{A}$.
2. $|\varphi(a^* b)| \leq \varphi(a^* a)\varphi(b^* b)$.
3. $\|\varphi\| = 1$.

Proposition 3.45.

The state space $S(\mathcal{A})$ of a unital C^*-algebra $\mathcal{A}$ is a convex subset of the unit ball of $\mathcal{A}^*$ which is closed in the weak-* topology. In particular, $S(\mathcal{A})$ is a weak-* compact subset of $\mathcal{A}^*$.

States on C^*-algebras

Proposition 3.46.

For every positive element a of a unital C^*-algebra $\mathcal{A}$, there exists a pure state $\varphi \in S(\mathcal{A})$ such that $\varphi a = \|a\|$.

Theorem 3.47.

The set of states of a unital C^*-algebra $\mathcal{A}$ separates the points of $\mathcal{A}$. That is, for every $a, b \in \mathcal{A}$ there exists $\varphi \in S(\mathcal{A})$ such that $\varphi a \neq \varphi b$.

Pure states

Definition 3.48.
A state φ of a unital C^*-algebra $\mathcal{A}$ is said to be pure if it is an extremal point of $S(\mathcal{A})$. Otherwise, φ is said to be mixed.

Definition 3.49.
Let H be a Hilbert space. A state φ of $B(H)$ is said to be a vector state if there exists a (unit) vector $\xi \in H$ such that

$$\varphi a = \langle \xi, a\xi \rangle, \quad \forall a \in \mathcal{A}.$$

Proposition 3.50.
Every vector state of $B(H)$ is pure.

Projections

Definition 3.51.
An element a of a *-algebra $\mathcal{A}$ is said to be a projection if $a = a^* = a^2$.

Proposition 3.52.
For a C^*-algebra $\mathcal{A}$, the projections are the extremal points of the positive cone $\mathcal{A}_+$.

Projection-valued measures

Definition 3.53.
Let (X, Σ) be a measurable space and H a Hilbert space. A map $E : \Sigma \to B(H)$ is said to be a projection-valued measure (PVM) if the following hold:

1. For every $S \in \Sigma$, $E(S)$ is a projection.
2. $E(\emptyset) = 0$.
3. $E(X) = I$.
4. For every countable collection $\{S_0, S_1, \ldots\}$ of pairwise-disjoint sets $S_j \in E$ and $f \in H$, we have $E(\bigcup_{j=0}^{\infty} S_j)f = \sum_{j=0}^{\infty} E(S_j)f$.

Projection-valued measures

Proposition 3.54.

Let (X, Σ) be a measurable space, H a Hilbert space, and $E : \Sigma \to B(H)$ a projection-valued map such that $E(X) = I$. Then, the following are equivalent:

1. *E is a PVM.*
2. *For every countable collection $\{S_0, S_1, \ldots\}$ of pairwise-disjoint sets $S_j \in E$, $\sum_{j=0}^{J} E_j$ converges as $J \to \infty$ in the weak operator topology.*
3. *For any two disjoint sets S and T, $E(S)E(T) = 0$.*

Projection-valued measures

Given a PVM $E : \Sigma \to B(H)$ and elements $\eta, \xi \in H$ we have:

- $E_{\eta,\xi} : \Sigma \to \mathbb{C}$ with $E_{\eta,\xi}(S) = \langle \eta, E(S)\xi \rangle$ is a finite complex measure.
- $E_\eta : \Sigma \to \mathbb{R}$ with $E_\eta(S) = E_{\eta,\eta}(S) = \langle \eta, E(S)\eta \rangle$ is a probability measure.

Spectral integrals

Theorem 3.55.
Given a PVM $E : \mathcal{B}(\mathbb{C}) \to B(H)$ and a bounded Borel-measurable function $f : \mathbb{C} \to \mathbb{C}$, there exists a unique operator $a \in B(H)$ such that

$$\langle \eta, a\xi \rangle = \int_{\mathbb{C}} f(\lambda)\, dE_{\eta,\xi}(\lambda).$$

Symbolically, we write

$$a = E(f) = \int_{\mathbb{C}} f(\lambda)\, dE(\lambda).$$

Spectral theorem

Theorem 3.56.
Let $a \in B(H)$ be a normal operator. Then, there exists a unique PVM $E : \mathcal{B}(\mathbb{C}) \to B(H)$, supported on the spectrum $\sigma(a) \subset \mathbb{C}$ such that

$$a = \int_{\mathbb{C}} \lambda \, dE(\lambda).$$

Remark.
If $f : \mathbb{C} \to \mathbb{C}$ is continuous on $\sigma(a)$, then $E(f)$ is identical to $f(a)$ as defined via the continuous functional calculus.

W^*-algebras

Definition 3.57.
A W^*-algebra (or abstract von Neumann algebra) $\mathcal{A}$ is a C^*-algebra that has a predual as a Banach space, i.e., we have $A = (\mathcal{A}_*)^*$ for a Banach space $\mathcal{A}_*$.

In addition to the norm and weak topologies, a W^*-algebra has the weak-* topology induced from the predual.

Definition 3.58.

- A linear map $T : \mathcal{A} \to \mathcal{B}$ between W^*-algebras $\mathcal{A}, \mathcal{B}$ is said to be normal if it is weak-* continuous.
- Correspondingly, a state $\varphi : \mathcal{A} \to \mathbb{C}$ of a W^*-algebra is called normal if there is $\rho \in \mathcal{A}_*$ such that

$$\varphi a = a\rho, \quad \forall a \in \mathcal{A}.$$

Commutants

Definition 3.59.
Let $\mathcal{A}$ be an algebra. The commutant of a set $X \subseteq \mathcal{A}$, denoted as X', is the set elements of $\mathcal{A}$ that commute with every element of X, i.e.,

$$X' = \{a \in \mathcal{A} : \ ax = xa, \forall x \in X\}.$$

The bicommutant of X, denoted as X'', is the commutant of X'.

Proposition 3.60.
With notation as above, the following hold.

- *X' is a subalgebra of $\mathcal{A}$.*
- *If $\mathcal{A}$ is unital, then X' is unital.*
- *If $\mathcal{A}$ is a *-algebra, then X' is a *-algebra.*
- *$X \subseteq X''$.*
- *$X''' = X'$.*

W^*-algebras

Theorem 3.61.
The set of projections of a W^*-algebra $\mathcal{A}$ spans a norm-dense subspace of $\mathcal{A}$.

Definition 3.62.
A W^*-algebra is said to be separable if it admits a faithful, normal representation on a separable Hilbert space H.

Proposition 3.63.
If a the predual $\mathcal{A}_*$ of a W^*-algebra $\mathcal{A}$ is separable in the norm topology, then $\mathcal{A}$ is separable.

Proposition 3.64.
If a W^*-algebra is infinite-dimensional, then it is non-separable in the norm topology.

Von Neumann algebras

Definition 3.65.
Let H be a Hilbert space. A (concrete) von Neumann algebra is a *-subalgebra of $B(H)$ which is closed in the weak operator topology.

Theorem 3.66 (von Neumann).
Let H be a Hilbert space and M a unital *-subalgebra of $B(H)$. Then, the following are equivalent:

1. *M is a von Neumann algebra.*
2. *M is closed in the strong operator topology.*
3. $M = M''$.

Von Neumann algebras

Theorem 3.67 (Sakai).

Every von Neumann algebra has a predual, and is thus a W^*-algebra. Moreover, the predual is unique up to isometric isomorphism.

Theorem 3.68.

Every abelian von Neumann algebra is isometrically isomorphic to $L^\infty(\mu)$ for some measure space (X, Σ, μ).

Analogously to our interpretation of the study of C^*-algebras as "non-commutative topology", we can interpret the study of von Neumann algebras as "non-commutative measure theory".

Further reading

[1] W. Arveson, *An Invitation to C^*-Algebras* (Graduate Texts in Mathematics). New York: Springer-Verlag, 1976, vol. 39.

[2] G. J. Murphy, *C^*-Algebras and Operator Theory*. Boston: Academic Press, 1990.

[3] M. Takesaki, *Theory of Operator Algebras I* (Encyclopaedia of Mathematical Sciences). Berlin: Springer, 2001, vol. 124.

Section 4

Embedding dynamical systems in operator algebras

Dirac–von Neumann axioms of quantum mechanics

1. States are density operators, i.e., positive, trace-class operators $\rho : H \to H$ on a Hilbert space H, with $\operatorname{tr} \rho = 1$.
2. Observables are self-adjoint operators, $a : D(a) \to H$.
3. Measurement expectation and probability:

$$\mathbb{E}_\rho a = \operatorname{tr}(\rho a), \quad \mathbb{P}_\rho(\Omega) = \mathbb{E}_\rho(E(\Omega)), \quad a = \int_{\mathbb{R}} \lambda \, dE(\lambda).$$

4. Unitary dynamics between measurements:

$$\rho_t = U^{t*} \rho_0 U^t.$$

5. Projective measurement:

$$\rho|_e = \frac{\sqrt{e}\rho\sqrt{e}}{\operatorname{tr}(\sqrt{e}\rho\sqrt{e})}, \quad 0 < e \leq I.$$

Algebraic formulation: States and observables

1. Associated with a physical system is a unital C^*-algebra $\mathcal{A}$.
2. The set of states of the system is the state space $S(\mathcal{A})$ of $\mathcal{A}$.
3. The set of observables of the system is the set of self-adjoint elements $\mathcal{A}_{\mathsf{sa}}$ of $\mathcal{A}$.
4. The set of values that can be obtained in a measurement of $a \in A_{\mathsf{sa}}$ corresponds to the spectrum $\sigma(a) \subset \mathbb{R}$.
5. The expected value of a measurement of $a \in \mathcal{A}_{\mathsf{sa}}$ when the system is in state $\varphi \in S(\mathcal{A})$ is given by $\varphi(a)$.

Algebraic formulation: Events and measurement probabilities

- The set of events (or effects) that can be observed is the set of positive elements $e \in \mathcal{A}_+$ such that $0 \leq e \leq \mathbb{1}$. If the system is in state $\varphi \in S(\mathcal{A})$, the probability to observe e is given by $\varphi(e)$.
- Supposing, further, that $\mathcal{A}$ is a W^*-algebra, the measurement probability for a to take value in a set $S \in \mathcal{B}(\mathbb{R})$ is given by $\varphi(E(S))$, where $E : \mathcal{B}(\mathbb{R}) \to \mathcal{A}$ is the PVM satisfying $a = \int_{\mathbb{R}} \lambda \, dE(\lambda)$.

Completely positive maps

Notation.
Given a C^*-algebra $\mathcal{A}$, $\mathbb{M}_n(\mathcal{A})$ is the C^*-algebra of $n \times n$ matrices with entries in $\mathcal{A}$.

Definition 4.1.
Let $T : \mathcal{A} \to \mathcal{B}$ be a linear map between C^*-algebras $\mathcal{A}$ and $\mathcal{B}$. Given $n \in \mathbb{N}$, we say that the map $T^{(n)} : \mathbb{M}_n(\mathcal{A}) \to \mathbb{M}_n(\mathcal{B})$ defined as $T^{(n)}([a_{ij}]) = [T(a_{ij})]$ is a matrix amplification of T.

Definition 4.2.
A linear map $T : \mathcal{A} \to \mathcal{B}$ between C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ is said to be:

- n-positive if $T^{(n)}$ is positive.
- Completely positive if it is n-positive for every $n \in \mathbb{N}$.

Completely positive maps

Theorem 4.3 (Stinespring).

Let $\mathcal{A}$ be a C^*-algebra and H a Hilbert space. A linear map $T : \mathcal{A} \to B(H)$ is completely positive iff there is a Hilbert space K, a representation $\pi : \mathcal{A} \to B(K)$ and a bounded linear map $V : K \to H$ such that

$$Ta = V(\pi a)V^*, \quad \forall a \in \mathcal{A}.$$

Proposition 4.4.

With notation as above, if $\mathcal{A}$ is abelian then $T : \mathcal{A} \to B(H)$ is completely positive iff it is positive.

Theorem 4.5 (Choi).

Let K and H be finite-dimensional Hilbert spaces of dimension m and n, respectively. Then, any completely positive map $T : B(K) \to B(H)$ take s the form $T(a) = \sum_{i=1}^{mn} V_i a V_i^*$ for some operators $V_i : K \to H$.

Quantum operations, quantum channels

Definition 4.6.
A linear map $T : \mathcal{B} \to \mathcal{A}$ between unital C^*-algebras $\mathcal{B}$ and $\mathcal{A}$ is said to be a quantum operation if:

1. T is completely positive.
2. $T\mathbb{1}_{\mathcal{B}} \leq \mathbb{1}_{\mathcal{A}}$.

If $T\mathbb{1}_{\mathcal{B}} = \mathbb{1}_{\mathcal{A}}$, T is said to be a quantum channel.

Proposition 4.7.
If $T : \mathcal{B} \to \mathcal{A}$ is a quantum operation, then for every state $\omega \in S(\mathcal{A})$ $T^*\omega \in \mathcal{B}^*$ is a positive functional satisfying $(T^*\omega)\mathbb{1}_B \leq 1$. Moreover, if T is a quantum channel, $(T^*\omega)\mathbb{1}_B = 1$.

Corollary 4.8.
The adjoint $T^* : \mathcal{A}^* \to \mathcal{B}^*$ of a quantum channel $T : \mathcal{B} \to \mathcal{A}$ maps the state space $S(\mathcal{A})$ into $S(\mathcal{B})$.

Quantum operations, quantum channels

Proposition 4.9.

A normal (weak-* continuous) quantum operation $T : \mathcal{B} \to \mathcal{A}$ between W^*-algebras $\mathcal{B}$ and $\mathcal{A}$ has a predual, i.e., $T = (T_*)^*$ for a unique linear map $T_* : \mathcal{A}_* \to \mathcal{B}_*$.

Algebraic formulation of measure-preserving dynamics

State space dynamics

$$\Phi : \Omega \to \Omega$$

$$\Phi_* : \mathcal{M}(\Omega) \to \mathcal{M}(\Omega), \quad \Phi_* \alpha = \alpha \circ \Phi^{-1}$$

- Φ: Invertible measure-preserving map.
- $\mathcal{M}$: Space of Borel measures on Ω.
- Φ_*: Pushforward map on measures.
- μ: Invariant probability measure, $\Phi_* \mu = \mu$.

Algebraic formulation of measure-preserving dynamics

Abelian formulation

$$U : \mathcal{A} \to \mathcal{A}, \quad Uf = f \circ \Phi$$
$$P : S_*(\mathcal{A}) \to S_*(\mathcal{A}), \quad Pp = p \circ \Phi^{-1}$$

- $\mathcal{A} = L^\infty(\mu)$: Abelian von Neumann algebra.
- $\mathcal{A}_{\text{sa}} = \{f \in \mathcal{A} : \ f \text{ is real-valued}\}$: Classical observables.
- $U : \mathcal{A} \to \mathcal{A}$: Koopman operator.
- $\mathcal{A}_* = L^1(\mu)$: Predual.
- $S_*(\mathcal{A}) = \{p \in \mathcal{A}_* : \ p \geq 0, \int_\Omega p \, d\mu = 1\}$: Probability densities.
- $\mathbb{E}_p : \mathcal{A} \to \mathbb{C}$ with $p \in S_*(\mathcal{A})$: Normal states, $\mathbb{E}_p f = \int_\Omega fp \, d\mu$.
- $P : S_*(\mathcal{A}) \to S_*(\mathcal{A})$: Transfer operator.

Algebraic formulation of measure-preserving dynamics

Non-abelian formulation

$$\mathcal{U} : \mathcal{B} \to \mathcal{B}, \quad \mathcal{U} a = U a U^*$$
$$\mathcal{P} : \mathcal{S}_*(\mathcal{B}) \to S_*(\mathcal{B}), \quad \mathcal{B} \rho = U^* \rho U$$

- $H = L^2(\mu)$: Hilbert space.
- $U : H \to H$: Unitary Koopman operator, $Uf = f \circ \Phi$.
- $\mathcal{B} = B(H)$: Non-abelian von Neumann algebra.
- $\mathcal{B}_{\text{sa}} = \{a \in \mathcal{B} : \ a \text{ is self-adjoint}\}$: Quantum observables.
- $\mathcal{U} : \mathcal{B} \to \mathcal{B}$: Induced Koopman operator.
- $\mathcal{B}_* = B_1(H)$: Predual.
- $S_*(\mathcal{B}) = \{\rho \in \mathcal{B}_* : \ \rho \geq 0, \operatorname{tr} \rho = 1\}$: Density operators.
- $\mathbb{E}_\rho : \mathcal{B} \to \mathbb{C}$ with $\rho \in \mathcal{S}_*(\mathcal{B})$: Normal states, $\mathbb{E}_\rho a = \operatorname{tr}(a\rho)$.
- $\mathcal{P} : S_*(\mathcal{B}) \to S_*(\mathcal{B})$: Induced transfer operator.

Algebraic formulation of measure-preserving dynamics

Classical–quantum consistency

Proposition 4.10.

The maps $U : \mathcal{A} \to \mathcal{A}$ and $\mathcal{U} : \mathcal{B} \to \mathcal{B}$ are quantum channels. Moreover, the following diagrams commute for the injective maps $\pi : \mathcal{A} \to \mathcal{B}$ and $\Gamma : S_*(\mathcal{A}) \to S_*(\mathcal{B})$:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "U"] \arrow[d, "\pi"'] & \mathcal{A} \arrow[d, "\pi"] \\ \mathcal{B} \arrow[r, "\mathcal{U}"] & \mathcal{B} \end{tikzcd} \qquad \begin{tikzcd} S_*(\mathcal{A}) \arrow[r, "P"] \arrow[d, "\Gamma"'] & S_*(\mathcal{A}) \arrow[d, "\Gamma"] \\ S_*(\mathcal{B}) \arrow[r, "\mathcal{P}"] & S_*(\mathcal{B}) \end{tikzcd}$$

- $\pi : \mathcal{A} \to \mathcal{B}$: Regular representation, $\pi f = a$ with $ag = fg$ for all $g \in H$.
- $\Gamma : S_*(\mathcal{A}) \to S_*(\mathcal{B})$: Mapping of probability densities into pure quantum states, $\Gamma(\pi) = \langle \sqrt{p}, \cdot \rangle \sqrt{p}$.

Section 5

Spectral theory of one-parameter evolution groups

Setting and objectives

General assumptions

- $\Phi : G \times \Omega \to \Omega$: Continuous-time, continuous flow on compact, metrizable space Ω.
- μ: Ergodic invariant Borel probability measure.
- $X : \Omega \to \mathbb{X}$ continuous observation map into metric space $\mathcal{X}$.
- $U^t : \mathcal{F} \to \mathcal{F}$: Koopman operator on Banach space $\mathcal{F}$ of complex-valued observables.

Given. Time-ordered samples

$$x_n = X(\omega_n), \quad \omega_n = \Phi^{t_n}(\omega_0), \quad t_n = (n-1)\,\Delta t.$$

Goal. Using the data x_n, identify a collection of observables $\zeta_j : \Omega \to \mathcal{Y}$ which have the property of evolving coherently under the dynamics in a suitable sense.

Setting and objectives

We recall the following facts from Section 2 (see Proposition 2.7 and Theorems 2.29, 2.30).

Theorem 5.1.

1. $\{U^t : C(\Omega) \to C(\Omega)\}_{t\in\mathbb{R}}$ *is a strongly continuous group of isometries.*
2. $\{U^t : L^p(\mu) \to L^p(\mu)\}_{t\in\mathbb{R}}$, $p \in [0,\infty)$ *is a strongly continuous group of isometries. Moreover,* $U^t : L^2(\mu) \to L^2(\mu)$ *is unitary.*
3. $\{U^t : L^\infty(\mu) \to L^\infty(\mu)\}_{t\in\mathbb{R}}$ *is a weak-** *continuous group of isometries.*

Notation.

- $\mathcal{F}$: Any of the $C(\Omega)$ or $L^p(\mu)$ spaces with $1 \le p \le \infty$.
- $\mathcal{F}_0$: Any of the $C(\Omega)$ or $L^p(\mu)$ spaces with $1 \le p < \infty$.
- C_0 (semi)group $\equiv$ strongly continuous (semi)group.
- C_0^* (semi)group $\equiv$ weak-* continuous (semi)group.

Generator of C_0 semigroups

Definition 5.2.
Let $\{S^t\}_{t\in\mathbb{R}}$ be a C_0 semigroup on a Banach space E. The generator $A : D(A) \to E$ of the semigroup $\{S^t\}_{t\geq 0}$ is defined as

$$Af = \lim_{t\to 0} \frac{S^t f - f}{t}, \quad f \in D(A),$$

where the limit is taken in the norm of E, and the domain $D(A) \subseteq E$ consists of all $f \in E$ for which the limit exists.

Generator of C_0 semigroups

Theorem 5.3.
With the notation of Definition 5.2, the following hold.

1. *A is closed and densely defined.*
2. *For all $f \in D(A)$ and $t \geq 0$, the function $t \mapsto S^t f$ is continuously differentiable, and satisfies*

$$\frac{d}{dt} S^t f = A S^t f = S^t A f.$$

3. *A uniquely characterizes the semigroup $\{S^t\}$, i.e., if $\{\tilde{S}^t\}$ is another C_0 semigroup on E with the same generator A, then $S^t = \tilde{S}^t$ for all $t \geq 0$.*

Generator of C_0^* semigroups

Definition 5.4.
Let $\{S^t\}_{t\geq 0}$ be a C_0^* semigroup on a Banach space E with predual E_*. The generator $A : D(A) \to E$ of the semigroup $\{S^t\}_{t\geq 0}$ is defined as the weak-* limit

$$\langle g, Af \rangle = \lim_{t\to 0} \frac{\langle g, S^t f - f \rangle}{t}, \quad f \in D(A), \quad \forall g \in E_*,$$

where the domain $D(A) \subseteq E$ consists of all $f \in E$ for which the limit exists.

Theorem 5.5.

With the notation of Definition 5.4, the following hold.

1. *A is weak-* closed and densely defined.*
2. *For all $f \in D(A)$ and $t \geq 0$, the function $t \mapsto S^t f$ is weak-* continuously differentiable, and satisfies*

$$\left\langle g, \frac{d}{dt} S^t f \right\rangle = \langle g, AS^t f \rangle = \langle g, S^t A f \rangle.$$

3. *A uniquely characterizes the semigroup $\{S^t\}$, i.e., if $\{\tilde{S}^t\}$ is another C_0^* semigroup on E with the same generator A, then $S^t = \tilde{S}^t$ for all $t \geq 0$.*

Generator of unitary C_0 groups

Theorem 5.6 (Stone).

Let $\{S^t\}_{t\geq 0}$ be a unitary C_0 group on a Hilbert space H. Then, the generator $A : D(A) \to H$ is skew-adjoint, i.e.,

$$A^* = -A.$$

Conversely, if $A : D(A) \to H$ is skew-adjoint, it is the generator of a unitary evolution group.

Generator of Koopman evolution groups

Corollary 5.7.

Under our general assumptions the following hold:

1. *The Koopman evolution groups $U^t : \mathcal{F}_0 \to \mathcal{F}_0$ are uniquely characterized by their generator $V : D(V) \to \mathcal{F}_0$, where*

$$Vf = \lim_{t \to 0} \frac{U^t f - f}{t}.$$

   Moreover, for $\mathcal{F}_0 = L^2(\mu)$, V is skew-adjoint.

2. *The Koopman evolution group $U^t : L^\infty(\mu) \to L^\infty(\mu)$ is uniquely characterized by its generator $V : D(V) \to \mathcal{F}_0$, where*

$$Vf = \lim_{t \to 0} \frac{U^t f - f}{t}$$

   in weak-* sense.

Generator of Koopman evolution groups

Theorem 5.8 (ter Elst & Lemańczyk).

Let (Ω, Σ) be a compact metrizable space equipped with its Borel σ-algebra Σ. Let μ be a Borel probability measure on Ω and $U^t : L^2(\mu) \to L^2(\mu)$ a C_0 unitary evolution group with generator $V : D(V) \to L^2(\mu)$. Then, the following are equivalent.

1. *For every $t \in \mathbb{R}$ there exists a μ-a.e. invertible, measurable, and measure-preserving flow $\Phi^t : \Omega \to \Omega$ such that $U^t f = f \circ \Phi^t$.*
2. *The space $\mathfrak{A}(V) = D(V) \cap L^\infty(\mu)$ is an algebra with respect to function multiplication, and V is a derivation on $\mathfrak{A}$:*

$$V(fg) = (Vf)g + f(Vg), \quad \forall f, g \in \mathfrak{A}(V).$$

Point spectrum

Definition 5.9.
Let $A : D(A) \to E$ be an operator on a Banach space with domain $D(A) \subseteq E$. The point spectrum of A, denoted as $\sigma_p(A) \subseteq \mathbb{C}$ is defined as the set of its eigenvalues. That is, $\lambda \in \mathbb{C}$ is an element of $\sigma_p(A)$ iff there is a nonzero vector $u \in E$ (an eigenvector) such that

$$Au = \lambda u.$$

Notation.

- We use the notation $\sigma_p(A; E)$ when we wish to make explicit the Banach space on which A acts.

Eigenvalues and eigenfunctions

Definition 5.10.
Let $A : D(A) \to E$ be the generator of a C_0 semigroup $\{S^t\}_{t\geq 0}$ on a Banach space E. We say that $\lambda \in \mathbb{C}$ is an eigenvalue of the semigroup if λ is an eigenvalue of A, i.e., there exists a nonzero $u \in D(A)$ such that

$$Au = \lambda u.$$

Lemma 5.11.
With notation as above, λ is an eigenvalue of $\{S^t\}$ if and only if z is an eigenvector of S^t for all $t \geq 0$, i.e., there exist $\Lambda^t \in \mathbb{C}$ such that

$$S^t u = \Lambda^t u, \quad \forall t \geq 0.$$

In particular, we have $\Lambda^t = e^{\lambda t}$.

Point spectra for measure-preserving flows

Theorem 5.12.
Let $\Phi^t : \Omega \to \Omega$ a be a measure-preserving flow of a probability space (Ω, Σ, μ). Let $U^t : L^p(\mu) \to L^p(\mu)$ be the associated Koopman operators on $L^p(\mu)$, $p \in [1, \infty]$, and $V : D(V) \to L^p(\mu)$ the corresponding generators. Then, the following hold for every $p, q \in [1, \infty]$ and $t \in \mathbb{R}$,

1. $\sigma_p(U^t, L^p(\mu)) = \sigma_p(U^t, L^q(\mu))$.
2. $\sigma_p(V, L^p(\mu)) = \sigma_p(V, L^q(\mu))$.
3. $\sigma_p(U^t)$ *is a subgroup of* S^1.
4. $\sigma_p(V)$ *is a subgroup of* $i\mathbb{R}$.

Corollary 5.13.
Every eigenfunction of V lies in $L^\infty(\mu)$, and thus in $L^p(\mu)$ for every $p \in [1, \infty]$.

Given $\lambda = i\alpha \in \sigma_p(V)$, we say that α is an eigenfrequency of V.

Generating frequencies

Definition 5.14.

Assume the notation of Theorem 5.12.

1. We say that $\{i\alpha_0, i\alpha_1, \ldots\} \subseteq \sigma_p(V)$ is a generating set if for every $i\alpha \in \sigma_p(V)$ there exist $j_1, j_2, \ldots, j_n \in \mathbb{Z}$ and $k_1, k_2, \ldots, k_n \in \mathbb{N}$ such that
$$\alpha = j_1\alpha_{k_1} + j_2\alpha_{k_2} + \ldots + j_n\alpha_{k_n}.$$
2. We say that $\sigma_p(V)$ is finitely generated if it has a finite generating set.
3. A generating set is said to be minimal if it does does not have any proper subsets which are generating sets.

Lemma 5.15.

1. *The elements of a minimal generating set are rationally independent.*
2. *If a minimal generating set has at least two elements, then $\sigma_p(V)$ is a dense subset of the imaginary line.*

Generating frequencies

Lemma 5.16.
Let $g_1, g_2, \ldots$ be eigenfunctions corresponding to the eigenvalues of the generating set in Definition 5.14, i.e., $Vg_j = i\alpha_j g_j$. Then, for every $i\alpha \in \sigma_p(V)$ with $\alpha = j_1\alpha_{k_1} + j_2\alpha_{k_2} + \ldots + j_n\alpha_{k_n}$,

$$z = g_{k_1}^{j_1} g_{k_2}^{j_2} \cdots g_{k_n}^{j_n}$$

is an eigenfunction of V corresponding to the eigenfrequency α.

Invariant subspaces

Notation.

- $H_p = \overline{\operatorname{span}\{u \in L^2(\mu) \colon u \text{ is an eigenfunction of } V\}}$.
- $H_c = H_p^\perp$.
- $\{z_0, z_1, \dots\}$: Orthonormal eigenbasis of H_p, $V z_j = i \alpha_j z_j$.

Theorem 5.17.
Let $\Phi^t : \Omega \to \Omega$ be a measure-preserving flow on a completely metrizable space with an invariant probability measure μ.

1. *H_p and H_c are U^t-invariant subspaces.*
2. *Every $f \in H_p$ satisfies*

$$U^t f = \sum_{j=0}^{\infty} \hat{f}_j e^{i\alpha_j t} z_j, \quad \hat{f}_j = \langle z_j, f \rangle_{L^2(\mu)}.$$

3. *Every $f \in H_c$ satisfies*

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T |\langle g, U^t f \rangle_{L^2(\mu)}| = 0, \quad \forall g \in L^2(\mu).$$

Pure point spectrum

Definition 5.18.
With the notation of Theorem 5.17, we say that a measure-preserving flow $\Phi^t : \Omega \to \Omega$ has pure point spectrum if $H_p = L^2(\mu)$.

Remark 5.19.
For a system with pure point spectrum:

1. The spectrum of V is not necessarily discrete.
2. The continuous spectrum is not necessarily empty.

Point spectra for ergodic flows

Proposition 5.20.

With the notation of Theorem 5.12, assume that $\Phi^t : \Omega \to \Omega$ is ergodic.

1. *Every eigenvalue $\lambda \in \sigma_p(V)$ is simple.*
2. *Every corresponding eigenfunction $z \in L^p(\mu)$ normalized such that $\|z\|_{L^p(\mu)} = 1$ for any $p \in [1, \infty]$ satisfies $|z| = 1$ μ-a.e.*

Factor maps

Definition 5.21.
Let $T_1 : \Omega_1 \to \Omega_1$ and $T_2 : \Omega_2 \to \Omega_2$ be measure-preserving transformations of the probability spaces $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$. We say that T_2 is a factor of T_1 if there exists a T_1-invariant set $S_1 \in \Sigma_1$ with $\mu_2(S_+1) = 1$, a T_2-invariant set $S_2 \in \Sigma_2$ with $\mu_2(S_2) = 1$, and a measure-preserving, surjective map $\varphi : S_1 \to S_2$ such that

$$T_2 \circ \varphi = \varphi \circ T_1.$$

Such a map φ is called a factor map and satisfies the following commutative diagram:

$$\begin{array}{ccc} M_1 & \xrightarrow{T_1} & M_1 \\ \varphi \downarrow & & \downarrow \varphi \\ M_2 & \xrightarrow{T_2} & M_2 \end{array} .$$

Metric isomorphisms

Definition 5.22.
With the notation of Definition 5.21, we say that T_1 and T_2 are measure-theoretically isomorphic or metrically isomorphic if there is a factor $\varphi : S_1 \to S_2$ with a measurable inverse.

Theorem 5.23 (von Neumann).
Let $\Phi^t : \Omega \to \Omega$ be a measure-preserving flow on a completely metrizable probability space (Ω, Σ, μ) with pure point spectrum. Then, Φ^t is metrically isomorphic to a translation on a compact abelian group $\mathcal{G}$. Explicitly, $\mathcal{G}$ can be chosen as the character group of the point spectrum $\sigma_p(V)$.

Metric isomorphisms

Corollary 5.24.

If $\sigma_p(V)$ is finitely generated, then Φ^t is metrically isomorphism to an ergodic rotation on the d-torus, where d is the number of generating frequencies of $\sigma_p(V)$. Explicitly, supposing that $\{i\alpha_1, \dots, i\alpha_d\}$ is a minimal generating set of $\sigma_p(V)$ with corresponding unit-norm eigenfunctions $z_1, \dots, z_d$ we have

$$R^t \circ \varphi = \varphi \circ \Phi^t,$$

where $R^t : \mathbb{T}^d \to \mathbb{T}^d$ is the torus rotation with frequencies $\alpha_1, \dots, \alpha_d$, and

$$\varphi(\omega) = (z_1(\omega), \dots, z_d(\omega)), \quad \mu\text{-a.e.}$$

Spectral isomorphisms

Definition 5.25.
With the notation of Definition 5.22, let $U_1 : L^2(\mu_1) \to L^2(\mu_1)$ and $U_2 : L^2(\mu_2) \to L^2(\mu_2)$ be the Koopman operators associated with T_1 and T_2, respectively. We say that T_1 and T_2 are spectrally isomorphic if there exists a unitary map $\mathcal{U} : L^2(\mu_1) \to L^2(\mu_2)$ such that

$$U_2 \circ \mathcal{U} = \mathcal{U} \circ U_1.$$

Theorem 5.26 (von Neumann).
Two measure-preserving flows with pure point spectra are metrically isomorphic iff they are spectrally isomorphic.

Section 6

Representation of classical dynamics in quantum circuits

Classical and quantum bits

- A (classical) bit is a pure state of the abelian algebra $\mathbb{C}^2$.
- A quantum bit, or qubit, is a pure state of the matrix algebra $B(\mathbb{C}^2) \simeq \mathbb{M}_2(\mathbb{C})$.
- Noisy classical bits and qubits are represented by mixed states of $\mathbb{C}^2$ and $\mathbb{M}_2$, respectively.

Quantum computers

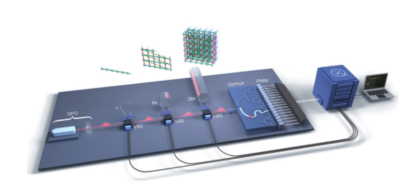
A quantum computer is a finite-dimensional quantum mechanical system associated with the tensor product Hilbert space $\mathbb{B}_n \equiv \mathbb{B}^{\otimes n}$ with $\mathbb{B} = \mathbb{C}^2$.

Notation.

- $|0\rangle$ and $|1\rangle$ are orthonormal basis vectors of $\mathbb{B}$ known as computational basis vectors.
- $|b_1 \cdots b_n\rangle \equiv |b_1\rangle \otimes \cdots \otimes |b_n\rangle$ are orthonormal basis vectors of $\mathbb{B}_n$.

Quantum computers

IBM Q One



AWS Borealis

Physical qubit implementations include superconducting charges, trapped ions, and photons.

Quantum circuits

A quantum circuit consists of wires, representing individual qubits, and gates representing operations (quantum channels) on qubits.

- The depth of a quantum circuit is the longest path in the circuit.

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A quantum circuit consists of wires, representing individual qubits, and gates representing operations (quantum channels) on qubits.

- The depth of a quantum circuit is the longest path in the circuit.

Goal. Given a C_0 group of unitary Koopman operators $U^t : H \to H$ induced by a measure-preserving flow with skew-adjoint generator $V : D(V) \to H$ and a subspace $H_L \subset D(V) \subset H$ of dimension 2^n, find a unitary $W : H_L \to \mathbb{B}_n$ such that e^{tG_L} with $G_L = W\Pi_L V \Pi_L W^*$ is representable by a circuit of low depth.