MATH 146
Current Problems in Applied Mathematics: Dynamical Systems and Data Science

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Section 1

Introduction

Ergodic theory

Ergodic theory studies the statistical behavior of measurable actions of groups or semigroups on spaces.

Definition 1.1.
A left action of a (semi)group G on a set Ω is a map $\Phi :$ $G \times \Omega \to \Omega$ with the following properties:

1. $\Phi(e, \omega) = \omega$, for the the identity element $e \in G$ and all $\omega \in \Omega$.
2. $\Phi(gh, \omega) = \Phi(g, \Phi(h, \omega))$, for all $g, h \in G$ and $\omega \in \Omega$.

   ↳ "g+h"

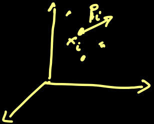
The set Ω is called the state space.

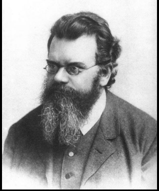
In this course, G will be an abelian group or semigroup that represents the time domain. Common choices include:

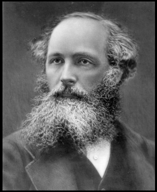
$$\overbrace{\mathbb{N}, \qquad \mathbb{Z}}^{\text{discrete time}}, \qquad \overbrace{\mathbb{R}_+, \qquad \mathbb{R}}^{\text{continuous time}}.$$

We write $\Phi^g \equiv \Phi(g, \cdot)$, $n \in \mathbb{N}, \mathbb{Z}$, and $t \in \mathbb{R}_+, \mathbb{R}$.

Ergodic theory




Ludwig Boltzmann


James Clerk Maxwell

$\Omega = \mathbb{R}^{6N}$ state spac

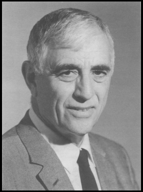
$f : \Omega \to \mathbb{R}$ observables

Ergodic theory has its origin in the mid 19th century with the work of Boltzmann and Maxwell on statistical mechanics.

The term ergodic is an amalgamation of the Greek words ergo (έργο), which means *work*, and odos (οδός), which means *street*.

Ergodic theory


George David Birkhoff


Bernard Osgood Koopman


John von Neumann

The mathematical foundations of the subject were established by Koopman, von Neumann, Birkhoff, and many others, in work dating to the 1930s.

Modern ergodic theory is a highly diverse subject with connections to functional analysis, harmonic analysis, probability theory, topology, geometry, number theory, and other mathematical disciplines.

Observables and ergodic hypothesis

Rather than studying the flow Φ directly, ergodic theory focuses on its induced action on linear spaces of observables, e.g.,

$$\mathcal{F} = \{f : \Omega \to \mathcal{Y}\},$$

for a vector space $\mathcal{Y}$ (oftentimes, $\mathcal{Y} = \mathbb{R}$ or $\mathbb{C}$).

$f, g \in \mathcal{F}$

$h = f + g \; : \; h(\omega) = f(\omega) + g(\omega)$

$g = \underset{\uparrow \text{scalar}}{c} f \; : \; g(\omega) = c \cdot f(\omega)$

Drawing on intuition from mechanical systems, Boltzmann postulated that time averages of observables should well-approximate expectation values with respect to a reference distribution, μ.

This is encapsulated in the ergodic hypothesis,

$$\underbrace{\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(\Phi^n(\omega))}_{\text{time average}} = \underbrace{\int_\Omega f \, d\mu}_{\text{space average}},$$

which is stipulated to hold for typical initial conditions $\omega \in \Omega$ and observables $f : \Omega \to \mathcal{Y}$ in a suitable class.

Operator-theoretic perspective

$U^g f = h \quad : h(\omega) = f(\phi^g(\omega))$

$U^g(f+h)(\omega) = (f+h)(\phi^g(\omega)) = f(\phi^g(\omega)) + h(\phi^g(\omega))$

$= (U^g f)(\omega) + (U^g h)(\omega)$

Definition 1.2.

1. For every $g \in G$, the composition operator, or Koopman operator is the linear map $U^g : \mathcal{F} \to \mathcal{F}$ defined as

$$U^g = f \circ \Phi^g.$$

2. The transfer operator $P^g : \mathcal{F}' \to \mathcal{F}'$ is the transpose of U^g, defined as

$$P^g \mu = \mu \circ U^g.$$

$= \{ \mu : \mathcal{F} \xrightarrow[\text{linear}]{} \mathbb{K} \}$

scalars

$\mathbb{R}, \mathbb{C}$

Koopman and transfer operators allow the study of nonlinear dynamics using techniques from linear operator theory.

A central theme of this course is that operator-theoretic techniques also provide a bridge between dynamical systems theory and data science.

$(P^g \mu) f = \mu(U^g f)$

$= \int (U^g f) \, d\mu = \int (f \circ \phi^g) \, d\mu =: \int f \, d\mu_*$

Connections with representation theory

$\phi : G \times \Omega \to \Omega$
$\phi^g = \phi(g, \cdot)$
$\phi^g \circ \phi^{g'} = \phi^{gg'}$

Observe that the set $\tilde{\Phi} = \{\Phi^g \mid g \in G\}$ equipped with composition of maps forms a group.

1. $h : G \to \tilde{\Phi}$ with $h(g) = \Phi^g$ is a group homomorphism.
2. $\varrho : \tilde{\Phi} \to \mathrm{End}(\mathcal{F})$ with $\varrho(\Phi^g) = U^g$ is a representation.

↑ observables

$U^g : \mathcal{F} \to \mathcal{F}, \quad U^g f = f \circ \phi^g$

Using operator-theoretic techniques, we study the dynamics through the induced representation $\rho : G \to \mathrm{End}(\mathcal{F})$, where $\rho = \varrho \circ h$:

↳ Endomorphisms ≡ Linear maps

$$\underbrace{\rho(\phi^g)}_{= U^g} \circ \underbrace{\rho(\phi^{g'})}_{= U^{g'}} = U^{gg'} = \rho(\phi^g \circ \phi^{g'})$$

$$\begin{tikzcd} G \arrow[r, "h"] \arrow[dr, "\rho"'] & \tilde{\Phi} \arrow[d, "\varrho"] \\ & \mathrm{End}(\mathcal{F}) \end{tikzcd}$$

Examples

Circle rotation in continuous time

Period: $\tau = 2\pi/\alpha$

Example 0

G is any group

Ω is any set

$\phi^g(\omega) = \omega$

$\Omega = S^1 = \{z \in \mathbb{C} : |z| = 1\}$

- $G = \mathbb{R}$, $\Omega = S^1$.
- Frequency parameter $\alpha \in \mathbb{R}$.
- $\Phi^t(\theta) = \theta + \alpha t \mod 2\pi$.

$f : S^1 \to \mathbb{R}$

$(U^t f)(\theta) = f(\phi^t(\theta)) = f(\theta + \alpha t \mod 2\pi)$

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(\phi^t(\omega))\,dt = \lim_{T\to\infty}\Bigg(\underbrace{\frac{1}{T}\int_0^{\tau N_T} f(\phi^t(\omega))\,dt}_{\frac{N_T}{T}\int_0^{\tau} f(\phi^t(\omega))\,dt} + \frac{1}{T}\int_{\tau N_T}^{T} f(\phi^t(\omega))\,dt \Bigg)$$

$\to 0$

$N_T = \left\lfloor \frac{T}{\tau} \right\rfloor$

$$= \frac{N_T}{T}\int_0^{2\pi} f(\omega + \theta \mod 2\pi)\underbrace{\frac{\tau}{2\pi}\,d\theta}_{d\mu}$$

$\theta = \alpha t$

$t = 0 \Rightarrow \theta = 0$

$t = \tau \Rightarrow \theta = 2\pi$

$dt = \frac{1}{\alpha} d\theta = \frac{\tau}{2\pi} d\theta$

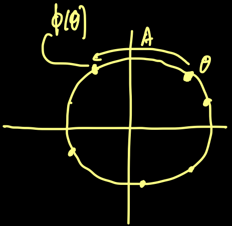
$$\longrightarrow \int_0^{2\pi} f(\omega + \theta \mod 2\pi)\,d\mu(\theta) = \text{space average}$$

$T \to \infty$, $\frac{N_T \tau}{T} \to 1$

Examples

Rational circle rotation in discrete time

- $G = \mathbb{Z}$, $\Omega = S^1$.
- Rotation angle $A \in [0, 2\pi)$, $A/(2\pi) \in \mathbb{Q}$.
- $\Phi^1(\theta) \equiv \Phi(\theta) = \theta + A \mod 2\pi$.



$A = \frac{p}{q} 2\pi$

$\phi(\theta) = \left(\theta + \frac{p}{q} 2\pi\right) \bmod 2\pi$

$\phi^q(\theta) = (\theta + p\, 2\pi) \bmod 2\pi$

$= \theta$

$\Rightarrow \phi$ is periodic with period $\leq q$

Examples

Irrational circle rotation in discrete time

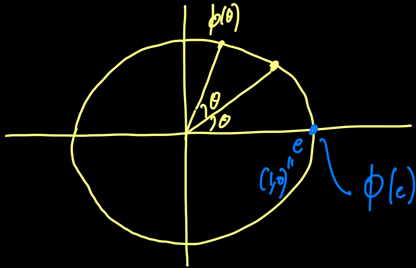
- $G = \mathbb{Z}$, $\Omega = S^1$.
- Rotation angle $A \in [0, 2\pi)$, $A/(2\pi) \notin \mathbb{Q}$.
- $\Phi^1(\theta) \equiv \Phi(\theta) = \theta + A \mod 2\pi$.

Examples

Doubling map

- $G = \mathbb{N}$, $\Omega = S^1$.
- $\Phi(\theta) = 2\theta \mod 2\pi$.

$\rightsquigarrow \; \phi : z \longmapsto z^2$, $z = e^{i\theta}$, $z^2 = e^{2i\theta}$



$\phi(e) = e$

- If θ is a rational multiple of 2π, the trajectory $\theta, \phi(\theta), \phi^2(\theta), \ldots$ is periodic
- If θ is not rational, the trajectory is not periodic

Examples

Rational torus flow

- $G = \mathbb{R}$, $\Omega = \mathbb{T}^2$.
- Frequency vector $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$, $\alpha_1/\alpha_2 \in \mathbb{Q}$.
- $\Phi^t(\theta) = \theta + \alpha t \mod 2\pi$.

Examples

Irrational torus flow

- $G = \mathbb{R}$, $\Omega = \mathbb{T}^2$.
- Frequency vector $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$, $\alpha_1/\alpha_2 \notin \mathbb{Q}$.
- $\Phi^t(\theta) = \theta + \alpha t \mod 2\pi$.

Examples

Arnold cat map

- $G = \mathbb{Z}$, $\Omega = \mathbb{T}^2$.
- $\Phi(\theta) = A\theta \mod 2\pi$, $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

Examples

Lorenz 63 system

- $G = \mathbb{R}$, $\Omega = \mathbb{R}^3$.
- $\Phi^t(x)$ is the solution map of the initial-value problem

$$\dot{x}(t) = v(x(t)), \quad x(0) = x$$

with

$$v(y) = (v_1, v_2, v_3)$$
$$v_1 = \sigma(x_2 - x_1), \quad v_2 = x_1(\rho - x_3), \quad v_3 = x_1 x_2 - \beta x_3,$$
$$\rho = 28, \sigma = 10, \quad \beta = 8/3.$$

Dynamical systems and data science

$$\begin{tikzcd} \Omega \arrow[r, "Y_t"] \arrow[d, "X"'] & \mathcal{Y} \\ \mathcal{X} \arrow[ur, dashed, "Z_t"'] & \end{tikzcd}$$

Given. Time-ordered samples $(x_0, y_0), (x_1, y_1), \dots, (x_{N-1}, y_{N-1})$ of observables $X : \Omega \to \mathcal{X}$ (covariate) and $Y : \Omega \to \mathcal{Y}$ (response), where $\mathcal{Y}$ is a vector space and

$$x_n = X(\omega_n), \quad y_n = Y(\omega_n), \quad \omega_n = \Phi^{t_n}(\overset{\omega}{\cancel{x}}_0), \quad t_n = (n-1)\,\Delta t.$$

Problem 1 [forecasting]. Using the data (x_n, y_n), construct ("learn") a function $Z_t : \mathcal{X} \to \mathcal{Y}$ that predicts Y at a lead time $t \geq 0$. That is, Z_t should have the property that $Z_t \circ X$ is closest to $Y_t := Y \circ \Phi^t$ among all functions in a suitable class.

Problem 2 [coherent pattern extraction]. Using the data x_n, identify a collection of observables $\varphi_j : \Omega \to \mathcal{Y}$ which have the property of evolving coherently under the dynamics in a suitable sense.

Dynamical systems and data science

In this course, we explore various approaches for pointwise approximation (for Problem 1) and spectral analysis (for Problem 2) of Koopman/transfer operators.

A major requirement is that the approximations are refinable, i.e., the learned functions Z_t and φ_j should have well-controlled limits as $N \to \infty$.

Challenges. Linear operators on infinite-dimensional function spaces can exhibit qualitatively new features which are not present in finite-dimensional linear algebra, including:

1. Discontinuous (unbounded) linear maps.
2. Elements of the spectrum which are not eigenvalues (e.g., continuous spectrum).

Further reading

[1] T. Berry, D. Giannakis, and J. Harlim, "Bridging data science and dynamical systems theory," *Notices Amer. Math. Soc.*, vol. 67, no. 9, pp. 1336–1349, 2020. DOI: 10.1090/noti2151.

[2] F. Cucker and S. Smale, "On the mathematical foundations of learning," *Bull. Amer. Math. Soc.*, vol. 39, no. 1, pp. 1–49, 2001. DOI: 10.1090/S0273-0979-01-00923-5.

[3] M. Dellnitz and O. Junge, "On the approximation of complicated dynamical behavior," *SIAM J. Numer. Anal.*, vol. 36, p. 491, 1999. DOI: 10.1137/S0036142996313002.

[4] T. Eisner, B. Farkas, M. Haase, and R. Nagel, *Operator Theoretic Aspects of Ergodic Theory*, ser. Graduate Texts in Mathematics. Springer, 2015, vol. 272.

[5] B. O. Koopman, "Hamiltonian systems and transformation in Hilbert space," *Proc. Natl. Acad. Sci.*, vol. 17, no. 5, pp. 315–318, 1931. DOI: 10.1073/pnas.17.5.315.

Further reading

[6] I. Mezić, "Spectral properties of dynamical systems, model reduction and decompositions," *Nonlinear Dyn.*, vol. 41, pp. 309–325, 2005. DOI: 10.1007/s11071-005-2824-x.