Fun with Curves

Problem 1. Let $\mathbf{c} : [a, b] \to \mathbb{R}^3$ be a C^1 curve and assume that $\mathbf{c}'(t) \neq \mathbf{0}$ for all t. Let $s(t)$ denote the arc-length function of $\mathbf{c}$:

$$s(t) = \int_a^t ||\mathbf{c}'(u)||\, du.$$

a. Using the fundamental theorem of calculus, show that $s(t)$ is differentiable and that

$$s'(t) = ||\mathbf{c}'(t)||.$$

Conclude that $s'(t) > 0$ for all t.

Recall the following theorem from one variable calculus (which is essentially the inverse function theorem for single variable functions!): If $f : [a, b] \to \mathbb{R}$ is differentiable and $f'(x) > 0$ for all x, then f^{-1} exists and is differentiable with derivative given by

$$\frac{d}{dy} f^{-1}(y) = \frac{-1}{df/dx}$$

By part (a), the arc-length function s satisfies the hypotheses of this theorem.

b. Let $\mathbf{d}(u) = \mathbf{c}(s^{-1}(u))$, for $u \in [0, s(b)]$; $\mathbf{d}(u)$ is called the *arc-length reparametrization* of $\mathbf{c}(t)$. Using the theorem above, show that

$$||\mathbf{d}'(u)|| = 1.$$

c. Show that the arc-length function of $\mathbf{d}$ is given by

$$s^*(u) = u.$$

Why does this justify calling $\mathbf{d}$ the arc-length reparametrization?

Problem 2. In this problem we investigate what might be called the mean value theorem for curves.

a. Let $\mathbf{c}(t) = (\cos(t^2), \sin(t^2))$ for $0 \leq t \leq \sqrt{\pi}$. Show that there is *no* t in $[0, \sqrt{\pi}]$ so that $\mathbf{c}(\sqrt{\pi}) - \mathbf{c}(0) = \sqrt{\pi}\mathbf{c}'(t)$.

Part (a) shows that if we are looking for a mean value theorem for curves, it *cannot* (in general) take the form

$$\mathbf{c}'(t) = \frac{\mathbf{c}(b) - \mathbf{c}(a)}{b - a} \tag{1}$$

as we might hope.

b. Show that we can't salvage equation (1) just by applying $||\cdot||$ to both sides, by considering the path $\mathbf{c}(t) = (\cos t, \sin t)$ with $0 \leq t \leq \pi$.

The best I have been able to prove in general is that if $\mathbf{c} : [a, b] \to \mathbb{R}$ is a C^1 path that does not cross itself, then there is a $t \in [a, b]$ so that

$$||\mathbf{c}'(t)|| \cos \theta_t = \frac{||\mathbf{c}(b) - \mathbf{c}(a)||}{b - a}$$

where θ_t is the angle between $\mathbf{c}'(t)$ and the vector $\mathbf{c}(t) - \mathbf{c}(a)$. The proof doesn't use anything that we don't know, but I won't go into it here. I have no idea how to interpret this result.

b. Let $\mathbf{c} : [a, b] \to \mathbb{R}$ be a C^1 path. Let $\ell(\mathbf{c})$ be the length of $\mathbf{c}$ and let $s(t)$ be the arc-length function of $\mathbf{c}$. Show that there is a $t \in [a, b]$ so that

$$||\mathbf{c}'(t)||(b - a) = \ell(\mathbf{c}).$$

[*Hint:* Use the mean value theorem for integrals and Problem 1(a).] Is it reasonable to call this a mean value theorem?