The Technicalities of Integration

Problem 1. Let $R = [a, b] \times [c, d]$ and let $f : R \to \mathbb{R}$, $g : R \to \mathbb{R}$ be integrable on R. In this exercise we'll work through the proof of the following fact: If k is any constant, then $kf + g$ is integrable on R and

$$\iint_R (kf + g)\, dA = k \iint_R f\, dA + \iint_R g\, dA. \tag{1}$$

a. Let n be a positive integer. Show that the associated Riemann sums satisfy

$$S_n(kf + g) = kS_n(f) + S_n(g).$$

b. Take limits on both sides of the result in part (a) to deduce that

$$\lim_{n\to\infty} S_n(kf + g) = k \lim_{n\to\infty} S_n(f) + \lim_{n\to\infty} S_n(g).$$

- c. Conclude that $kf + g$ is integrable on R and that the equality (1) holds.

Problem 2. Let

$$f(x, y) = \begin{cases} 1 & x \text{ rational} \\ 2y & x \text{ irrational.} \end{cases}$$

Let's prove that this function is *not* integrable on $[0, 1] \times [0, 1]$.

Let n be a positive integer. Let

$$\begin{aligned} 0 = x_0 < x_1 < \quad \cdots \quad < x_{n-1} < x_n = 1 \\ 0 = y_0 < y_1 < \quad \cdots \quad < y_{n-1} < y_n = 1 \end{aligned}$$

be the associated regular partitions of $[0, 1]$. Let $R_{jk} = [x_j, x_{j+1}] \times [y_k, y_{k+1}]$. We will cleverly choose the $\mathbf{c}_{jk} \in R_{jk}$ to force the associated Riemann sums to have different limits.

a. Show that $x_j = j/n$ and $y_k = k/n$. Conclude that the x_j's and y_k's are all rational.

b. Let $\mathbf{c}_{jk} = (x_j, y_k)$. Show that the associated Riemann sum satisfies

$$S_n(f) = 1.$$

c. Now take $\mathbf{c}_{jk}^*$ as follows. If $k < n/2$ then let $\mathbf{c}_{jk}^* = (x_j^*, y_k)$ where x_j^* is an irrational number in $[x_j, x_{j+1}]$ (for example, $x_j^* = x_j + \sqrt{2}/2n$ works). If $k \geq 1/2$ then let $\mathbf{c}_{jk}^* = (x_j, y_k)$ Show that the associated Riemann sum is given by

$$S_n^*(f) = \sum_{0 \leq k < n/2} 2y_k \Delta y + \sum_{n/2 \leq k \leq n-1} \Delta y.$$

d. Show that

$$\lim_{n \to \infty} S_n^*(f) = \frac{3}{4}.$$

[*Hint:* Realize $S_n^*(f)$ as a Riemann sum for a certain *single* variable function.]

e. Conclude that

$$\lim_{n \to \infty} S_n(f) \neq \lim_{n \to \infty} S_n^*(f)$$

which shows that f is *not* integrable.