A Familiar Function Revisited

Throughout what follows, we let

$$T(u, v) = (u^2 - v^2, 2uv).$$

In a previous assignment we considered T to be a vector field, but now we want to study T as a map.

Problem 1. Determine the action of T on vertical lines as follows. Fix $\alpha \in \mathbb{R}$ and consider the vertical line $u = \alpha$. The points on this line all have the form (α, v). Determine the curve traced out by $T(\alpha, v)$ as v varies. You will need to consider the cases $\alpha = 0$ and $\alpha \neq 0$ separately.

Problem 2. Determine the action of T on horizonal lines as follows. Fix $\beta \in \mathbb{R}$ and consider the horizontal line $v = \beta$. The points on this line all have the form (u, β). Determine the curve traced out by $T(u, \beta)$ as u varies. You will need to consider the cases $\beta = 0$ and $\beta \neq 0$ separately.

Problem 3. Using the results of the preceeding exercises, find the image of $D^* = [0, 1] \times [0, 1]$ under T.

Let C_1 and C_2 be two differentiable curves in $\mathbb{R}^2$, intersecting at the point (α, β). The *angle between* C_1 *and* C_2 *at* (α, β) is defined to be the angle between the tangent vectors to C_1 and C_2 at the point (α, β).

Problem 4. Fix a point $(\alpha, \beta) \in \mathbb{R}^2$. Let C_1 and C_2 be two curves in the (u, v) plane that intersect at (α, β). Show that, unless $(\alpha, \beta) = (0, 0)$, then T *preserves the angles* at (α, β) by showing that the angle between the curves C_1 and C_2 at the point (α, β) is the same as the angle between the image curves $T(C_1)$ and $T(C_2)$ at the point $T(\alpha, \beta)$. [*Suggestion:* By writing C_1 and C_2 parametrically, use the chain rule to compute the tangent vectors to $T(C_1)$ and $T(C_2)$. Remember that the angle between two vectors is related to their dot product.]