The Integrals of Vector Calculus

This is just a short guide to the many integrals we have defined, indicating how the computation of each can be reduced to computing single variable integrals. Since they can also be useful for computations, I have included the statements of Green's, Stokes' and Gauss' theorems as well.

1 Double Integrals in the Plane

If $D \subset \mathbb{R}^2$ and $f : D \to \mathbb{R}$ is a continuous function, then the double integral is

$$\iint_D f\, dA.$$

By Fubini's Theorem, it may be computed using iterated single variable integrals. When $f \geq 0$, the integral represents the volume above D and below the graph of f. When $f = 1$, the integral equals the area of D.

2 Triple Integrals in 3-Space

If $W \subset \mathbb{R}^3$ and $f : W \to \mathbb{R}$ is a continuous function, the triple integral is

$$\iiint_W f\, dV.$$

By Fubini's Theorem, it is computed using iterated single variable integrals. When $f = 1$, the integral equals the volume of W.

3 Path Integrals

If $C \subset \mathbb{R}^3$ is a curve and $f : \mathbb{R}^3 \to \mathbb{R}$ is a continuous function, then the path integral of f along C is

$$\int_C f\, ds.$$

If $\mathbf{c} : [a, b] \to \mathbb{R}^3$ is a one-to-one C^1 parametrization of C then the path integral is given by

$$\int_C f\, ds = \int_a^b f(\mathbf{c}(t))\, \|\mathbf{c}'(t)\|\, dt$$

the right hand side being an ordinary single variable integral. If $f = 1$ then the integral is equal to the length of C.

We can also do path integrals in the plane. Here we have $f : \mathbb{R}^2 \to \mathbb{R}$ and C is an oriented curve in the plane given by $\mathbf{c} : [a, b] \to \mathbb{R}^2$. All of the above still hold, we just need to keep in mind that everything is now two-dimensional.

4 Line Integrals

If $C \subset \mathbb{R}^3$ is an oriented curve and $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is a continuous vector field, then the line integral of $\mathbf{F}$ along C is

$$\int_C \mathbf{F} \cdot d\mathbf{s}.$$

If $\mathbf{c} : [a, b] \to \mathbb{R}^3$ is a one-to-one orientation preserving C^1 path parametrizing C, then the line integral is given by

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt$$

whereas if $\mathbf{c}$ is orientation reversing we must insert a minus sign in front of the integral on the right. If $\mathbf{F} = \nabla f$ for some function f then

$$\int_C \mathbf{F} \cdot d\mathbf{s} = f(\mathbf{c}(b)) - f(\mathbf{c}(a)).$$

Line integrals are related to path integrals as follows. If $\mathbf{u}$ denotes the unit tangent vector to the curve C (i.e. $\mathbf{u}(t) = \mathbf{c}'(t)/||\mathbf{c}'(t)||$) then

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_C \mathbf{F} \cdot \mathbf{u}\, ds.$$

This is a path integral since $\mathbf{F} \cdot \mathbf{u}$ is a scalar at each point on the curve.

We can also do line integrals in the plane. Here we have $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ and C is an oriented curve in the plane given by $\mathbf{c} : [a, b] \to \mathbb{R}^2$. All of the above still holds, we just need to keep in mind that everything is now two-dimensional.

5 Surface Integrals

If S is a surface and $f : \mathbb{R}^3 \to \mathbb{R}$ is a continuous function, then the surface integral of f on S is

$$\int_S f\, dS.$$

If $\Phi : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ is a one-to-one C^1 parametrization of S, then the surface integral is given by

$$\int_S f\, dS = \iint_D f(\Phi(u, v))\, \|\mathbf{T}_u \times \mathbf{T}_v\|\, du\, dv$$

which can be computed as iterated single variable integrals. If $f = 1$ then the integral is equal to the surface area of S.

6 Flux Integrals

If S is an oriented surface and $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is a continuous vector field, then the flux integral of $\mathbf{F}$ on S is

$$\iint_S \mathbf{F} \cdot d\mathbf{S}.$$

If $\Phi : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ is a one-to-one orientation preserving C^1 parametrization of S, then the flux integral is given by

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\Phi(u, v)) \cdot (\mathbf{T}_u \times \mathbf{T}_v)\, du\, dv$$

and the right hand side can be computed as iterated single variable integrals. If Φ is orientation reversing, then we need to insert a minus sign in front of the integral on the right.

Flux integrals are related to surface integrals as follows. If $\mathbf{n}$ denotes the unit normal vector to the surface S then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS.$$

This is a surface integral since $\mathbf{F} \cdot \mathbf{n}$ is a scalar at each point on the surface.

7 Green's Theorem

Green's theorem relates line integrals in the plane to double integrals. If $\mathbf{F}(x, y) = (P(x, y), Q(x, y))$ is a C^1 vector field and D is any region in the plane that can be cut into pieces that are simultaneously x-simple and y-simple then

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{s} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$$

where the boundary ∂D of D is oriented so that the region D is on the left as you traverse ∂D.

8 Stokes' Theorem

Stokes' theorem relates line integrals around curves in $\mathbb{R}^3$ so certain flux integrals. If $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is a C^1 vector field and S is an oriented surface with boundary ∂S given the induced orientation, then

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}.$$

Green's theorem is the special case of Stokes' theorem in which the surface S is a subset of the plane.

9 Gauss' Theorem

Gauss' theorem, also known as the divergence theorem, relates flux integrals over closed surfaces to triple integrals over the regions that they contain. Specifically, if $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is a C^1 vector field, W is an elementary region in $\mathbb{R}^3$ (or a region that can be cut into elementary pieces) and ∂W is the surface bounding W given the outward orientation, then

$$\iint_{\partial W} \mathbf{F} \cdot d\mathbf{S} = \iiint_W \nabla \cdot \mathbf{F} \, dV.$$