Problem 19, Part 2

Problem 19b. Show that the function

$$v(x, y) = 3x^2y - y^3$$

is harmonic and satisfies the equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

These equations are known as the *Cauchy-Riemann equations*. Show also that for *any* pair of C^2 functions $u(x, y)$ and $v(x, y)$ satisfying the Cauchy-Riemann equations, both u and v are harmonic.

A pair of functions satisfying the Cauchy-Riemann equations are called *harmonic conjugates*. Such pairs of functions are very important in *complex analysis*, the subject concerned with the calculus of complex valued functions of a complex variable. An interesting fact regarding harmonic functions is the following: any pair of C^1 functions satisfying the Cauchy-Riemann equations must in fact be C^k for any k. That is, once differentiable functions satisfying the Cauchy-Riemann equations are actually infinitely differentiable!