More Matrices

Problem 1. Let A be an $m \times n$ matrix. Show that if $A\mathbf{x} = \mathbf{0}$ for every $\mathbf{x} \in \mathbb{R}^n$ then A is the zero matrix (that is, every entry in A is 0). *Hint:* What happens if you take $\mathbf{x} = \mathbf{e}_i$?

Problem 2. Let

$$B = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

be a symmetric 2×2 matrix with $a \neq 0$. As in the text, define the associated quadratic form by

$$\begin{aligned} Q(x,y) &= \begin{pmatrix} x & y \end{pmatrix} B \begin{pmatrix} x \\ y \end{pmatrix} && (1) \\ &= ax^2 + bxy + cy^2 && (2) \\ &= a\left(\left(x + \frac{b}{a}y\right)^2 + \left(\frac{ac - b^2}{a^2}\right) y^2\right). && (3) \end{aligned}$$

If $ac - b^2 < 0$ show that this form is *indefinite*; that is, show that Q takes on both positive and negative values. In fact, show that there are arbitrarily small points where Q is positive and arbitrarily small points where Q is negative. *Hint:* Try to find small points that make one of the two terms in (3) vanish.

Problem 3. Use Problem 2 and the Second Order Taylor Approximation to show that if $f : \mathbb{R}^2 \to \mathbb{R}$ is of class C^3, $\mathbf{x}_0$ is a critical point of f and the Hessian $Hf(\mathbf{x}_0)$ is indefinite, then f has a saddle point at $\mathbf{x}_0$. That is, show that under these hypotheses there are points $\mathbf{x}$ arbitrarily close to $\mathbf{x}_0$ for which $f(\mathbf{x}) > f(\mathbf{x}_0)$ and points $\mathbf{x}$ arbitrarily close to $\mathbf{x}_0$ for which $f(\mathbf{x}) < f(\mathbf{x}_0)$.