Math 14 Fall 2004

Calculus of Vector-valued Functions, Honors

First Project: Some Problems on Infinite Series

Due Date: Friday, November 5, 2004

Your name:

Instructions: Solve each of the 6 problems. You must justify all of your answers to receive credit.

You will need to use your own paper. Please write **neatly** on only one side of each sheet of paper.

The Honor Principle requires that you work on this project only with other students enrolled in this class. The only other person you may consult regarding this project is the course instructor. Each student is responsible for independently writing up his or her own solutions. No copying!

Let $a_1, a_2, a_3, \ldots$ be a sequence of real numbers. Since this sequence is infinite, what does it mean to add together all of its terms? That is, what meaning can we give to the quantity

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots ? \tag{1}$$

Such a sum is called an *infinite series.*

For any given n let

$$S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n.$$

This is called the n^{th} *partial sum* of the series (1). If the limit of partial sums

$$\lim_{n\to\infty} S_n$$

exists then we say that the infinite series (1) *converges* and we call the value of the limit the *sum* of the series. In this case we write

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i.$$

This should be interpreted as follows: as we add together more and more terms of the sequence, the finite sums tend to get closer and closer to a certain value, and this value we *define* to be the sum of the infinite series.

Problem 1. Show that if the series

$$\sum_{n=1}^{\infty} a_n$$

converges then

$$\lim_{n\to\infty} a_n = 0.$$

[*Hint:* Consider the difference $S_n - S_{n-1}$.]

Problem 2. Let $r \neq 1$ be a real number and let $a_n = r^{n-1}$. Show that

$$S_n = \frac{1 - r^n}{1 - r}.$$

[*Hint:* Write S_{n+1} in two different ways.]

Problem 3. Let r be a real number with $|r| < 1$. Using the result of the previous exercise show that

$$\sum_{n=1}^{\infty} r^{n-1} = \frac{1}{1 - r}.$$

This type of series is called a *geometric series.*

For the next problem we need the following

Theorem 1. *Suppose that $a_i \geq 0$ for all i and that there is a constant $C > 0$ so that*

$$S_n \leq C$$

for all n. Then $\lim_{n\to\infty} S_n$ exists and is $\leq C$. That is, the series

$$\sum_{n=1}^{\infty} a_n$$

converges and its sum is $\leq C$

Here's the idea of the proof. If $a_i \geq 0$ for all i then we have

$$S_1 \leq S_2 \leq S_3 \leq S_4 \leq \cdots \leq C.$$

since to get S_{n+1} from S_n we add a nonnegative quantity. That is, the partial sums are increasing but never exceed C. This forces them to converge.

Problem 4. Let $p > 1$. Show that the partial sums of the *p-series*

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

are all less than $p/(p-1)$ by comparing them to the improper integral

$$\int_1^{\infty} \frac{dx}{x^p}.$$

Conclude that the series converges to a number that is $\leq p/(p-1)$.

Problem 5. In this problem we will actually *compute* the value of the p-series when $p = 2$ by evaluating a certain improper integral in the plane.

a. Let

$$f(x,y) = \frac{1}{1-xy}$$

and let $D = [0,1] \times [0,1]$. Use Problem 3 to show that for $(x,y) \in D$, $(x,y) \neq (1,1)$ we have

$$f(x,y) = \sum_{n=1}^{\infty} (xy)^{n-1}.$$

b. Using the result from part (a), show that

$$\int_0^1 \int_0^1 \frac{1}{1-xy} dx\, dy = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

c. Let $T(u,v) = (u-v, u+v)$. Find the region D^* satisfying $T(D^*) = D$. Use this to make the change of variables $x = u - v$, $y = u + v$ in the integral in part (b). Use $dv\, du$ as the order of integration.

d. Evaluate the inner-most integrals in the result of part (c). This yields a pair of integrals with respect to u alone. Make the substitution $u = \sin\theta$ in these integrals and simplify.

e. Evaluate the integrals in part (d). Conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

[*Hint:* Try a few trigonometric identities to simplify the integrands down to linear polynomials.]

Problem 6. Imitating your work in parts (a) and (b) of Problem 5, show that

$$\int_0^1 \int_0^1 \int_0^1 \frac{1}{1 - xyz} dx\, dy\, dz = \sum_{n=1}^{\infty} \frac{1}{n^3}.$$

Unlike Problem 5, there's no known trick for evaluating the integral in Problem 6. In fact, no one currently knows the the exact value of either the integral or the sum of the series occurring in Problem 6! In general, when p is even, there *is* a formula for the sum of the p-series: it is always a rational multiple of π^p. However, the exact value of the sum of the p-series when p is odd is not known in any case.