Math 17 Winter 2018: Knot Theory
Fundamental group: summary and suggested problems

1. What are all the groups of order 4 (up to isomorphism)?

   Recall that for a topological space X with a point $x_0 \in X$ we defined the *fundamental group* $\pi_1(X, x_0)$ to be the group $(S, *)$ where:
   S is the set of paths on X starting and ending at x_0, where two paths are considered the same if they can be continuously deformed into each other;
   for paths a and b, $a * b$ is the path obtained by going along a, and then along b.

   We discussed in class that for any point $x_0 \in \mathbb{R}^2$, $\pi_1(\mathbb{R}^2, x_0)$ is the trivial group (i.e. all paths can be deformed into the constant path at x_0). We also discussed that for any point x_0 on the circle S^1, $\pi_1(S^1, x_0) \cong \mathbb{Z}$ (a path, up to continuous deformation, can be specified by choosing a direction along the circle, and a number of times to wind the path around the circle in that direction).

   Of course, we made these discoveries without a precise notion of a "topological space" or "continuous deformation". The only topological spaces we'll look at will be surfaces, knots, or complements of knots. Imagine the path being a very stretchy string, with a starting and ending point, that is entirely contained within the space X. Two paths are the same if we can deform one to look like the other, only by moving it and stretching it, but without cutting and without ever moving the starting or ending point. We "add" paths by tying the end of the first path to the beginning of the second path. The place where we tied is now in the middle of the new path, and is allowed to leave x_0 when we deform.

   For the following problems, pick any point x_0 you wish in the space in question.

2. What is $\pi_1(\mathbb{R}^3, x_0)$?

3. Let S^2 be the sphere. What is $\pi_1(S^2, x_0)$?

4. Let U be a slightly thickened unknot in $\mathbb{R}^3$ (e.g. imagine it is made out of a rope). Let $\mathbb{R}^3 \setminus U$ denote the space obtained by removing U from $\mathbb{R}^3$, i.e. all the points in $\mathbb{R}^3$ that are not on U. What is $\pi_1(\mathbb{R}^3 \setminus U, x_0)$?

5. Let T be a torus. What is $\pi_1(T, x_0)$? This is a tricky question. It might help to draw your torus "flat", as a square with its edges identified in pairs.