Math 19 final exam review problems - Answers

(1) $f_1 = g = x^2$, $f_2 = -x^2$.

(2) Suits only: $52!/(13!)^4$. Values only: $52!/(4!)^{13}$.

(3) $\sum_{k=10}^{20} \sum_{j=4}^{12} (j^2 2^k + 10) = 636(2^{21} - 2^{10}) + 990$.

(4) **Claim.** For every positive integer n, $n^2 + n$ is odd.
Problem: incorrect base case.

Claim. For each natural number n there is a natural number greater than n^2.
Problem: proving a universally quantified statement by example.

Claim. Define the relation R on $\mathbb{Z}$ by $R(n, m) \leftrightarrow [n - m$ is even$]$. Then R is an equivalence relation.
Problem: assumes the claim and unwraps the definition, rather than proving the claim by proving the pieces of the definition.

Claim. For real numbers a and b, $-b \leq a \leq b$ if and only if $|a| \leq b$.
Problem: proves only one direction of the bi-implication.

(5) Count each of the following situations with the appropriate permutation or combination.
(a) $P(8, 6)$
(b) $C(25, 8)$
(c) $C(32, 14)$
(d) $P(150, 15)$

(6) $C(21, 7) = P(21, 7)/7!$ and $P(21, 14)/14!$ and $C(21, 14)$. There are more convoluted ways as well, I'm sure, but they would involve multiplying and dividing many times to get to the correct fraction.

(7) $p(\text{line } 3|\text{defective}) = 0.2$.

(8) Partial answers given.
(a) $A = \{1, 2, 3, 4\}$. $R_1 = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$. $R_1 \cup R_2 =\leq$ relation. $R_1 \cap R_2 = \emptyset$.
(b) $A = \mathbb{Z}$. $R_1(x, y) = \{(x, y) : x = \pm y\}$. $R_1 = R_2 = R_1 \cup R_2 = R_1 \cap R_2$.
(c) $A = \{a, b, c, d\}$. $R_1 \cap R_2 = \emptyset$.

(9) (a) $A \subseteq B$

   (b) $x \leq y$
   (c) $x < y$
   (d) $x > y$
   (e) $x \geq y$

(10) (a) $C(8,3)$
   (b) $1/C(8,3)$
   (c) $1/(C(8,3) \cdot 3!) \; = \; 1/P(8,3)$

(11) (a) $C(4,2)/C(52,2)$
   (b) $[C(4,2)C(50,3) - C(4,2)C(48,3)]/[C(52,2)C(50,3)]$
   (c) Same as before.
   (d) Square of part (b).

(12) (a) 96%
   (b) 1/25 (yes, the match with (a) was an oversight)

(13) $E(X) = 0.8$, $V(X) = 1.46$.

(14) no

(15) $\sum_{i=2}^{10} \sum_{j=1}^{15} (20i - \frac{1}{2}j^3) = -48600.$

(16) kn^2

(17) (a) $(20!)/2^{10}$
   (b) $20!$
   (c) $(20!)/(2^{10}10!)$

(18) $4 \cdot 3[13C(13,4) + C(13,2)C(13,3)]$

(19) Prove that for all $n \geq 1$, $L_1^2 + L_2^2 + \ldots + L_n^2 = L_n L_{n+1} - 2$.
   Induction. Base case: $L_1 L_2 - 2 = 1 \cdot 3 - 2 = 1 = L_1^2$.
   Inductive hypothesis: for some $n \geq 1$, $L_1^2 + L_2^2 + \ldots + L_n^2 = L_n L_{n+1} - 2$.
   By the hypothesis, $L_1^2 + L_2^2 + \ldots + L_n^2 + L_{n+1}^2 = L_n L_{n+1} - 2 + L_{n+1}^2$.
   We need this to equal $L_{n+1} L_{n+2} - 2$. Note $L_n L_{n+1} + L_{n+1}^2 - 2 = L_{n+1}(L_n + L_{n+1}) - 2$ and apply the recursive definition.

(20) For $O(\frac{1}{100}x^3)$: $C = 100$, $k = 1$. For not $O(100x)$: let $x > 100C, k$.

(21) (a) $\exists x \forall y (x + y = 10)$ is true with domain $\{5\}$ and false in $\mathbb{N}$.
   (b) $\forall y \exists x (x + y = 10)$ is true in $\mathbb{Z}$ and false in $\mathbb{N}$.