> ***Product Rule***
>
> For events A and B with nonzero probabilities in a sample space S,
>
> $$P(A \cap B) = P(A)P(B|A) = P(B)P(A|B) \tag{2}$$
>
> and we can use either $P(A)P(B|A)$ or $P(B)P(A|B)$ to compute $P(A \cap B)$.

Example 3 **Consumer Survey** If 60% of a department store's customers are female and 75% of the female customers have charge accounts at the store, what is the probability that a customer selected at random is a female and has a charge account?

SOLUTION Let: S = all store customers
F = female customers
C = customers with a charge account

If 60% of the customers are female, then the probability that a customer selected at random is a female is

$$P(F) = .60$$

Since 75% of the female customers have charge accounts, the probability that a customer has a charge account, given that the customer is a female, is

$$P(C|F) = .75$$

Using equation (2), the probability that a customer is a female and has a charge account is

$$P(F \cap C) = P(F)P(C|F) = (.60)(.75) = .45$$

Matched Problem 3 If 80% of the male customers of the department store in Example 3 have charge accounts, what is the probability that a customer selected at random is a male and has a charge account?

❑ PROBABILITY TREES

We used tree diagrams in Section 6-1 to help us count the number of combined outcomes in a sequence of experiments. In much the same way we will now use probability trees to help us compute the probabilities of combined outcomes in a sequence of experiments. An example will help make clear the process of forming and using probability trees.

Example 4 **Probability Tree** Two balls are drawn in succession, without replacement, from a box containing 3 blue and 2 white balls (Fig. 4). What is the probability of drawing a white ball on the second draw?

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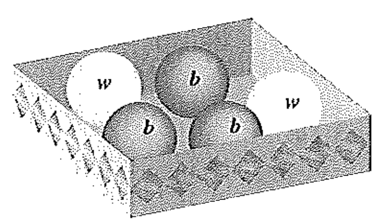


FIGURE 4

SOLUTION We start with a tree diagram (Fig. 5) showing the combined outcomes of the two experiments (first draw and second draw):

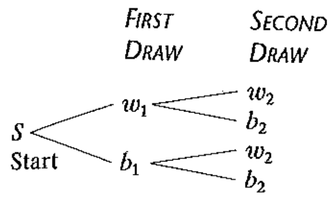


FIGURE 5

We now assign a probability to each branch on the tree (Fig. 6). For example, we assign the probability $\frac{2}{5}$ to the branch Sw_1, since this is the probability of drawing a white ball on the first draw (there are 2 white balls and 3 blue balls in the box). What probability should be assigned to the branch w_1w_2? This is the conditional probability $P(w_2|w_1)$, that is, the probability of drawing a white ball on the second draw given that a white ball was drawn on the first draw and not replaced. Since the box now contains 1 white ball and 3 blue balls, the probability is $\frac{1}{4}$. Continuing in the same way, we assign probabilities to the other branches of the tree and obtain Figure 6.

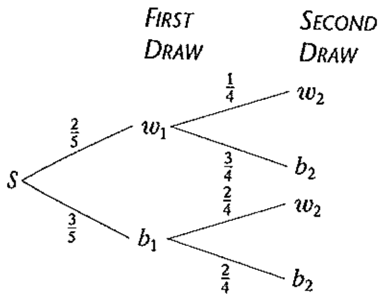


FIGURE 6

What is the probability of the combined outcome $w_1 \cap w_2$, that is, of drawing a white ball on the first draw and a white ball on the second draw?* Using the product rule (2), we have

$$\begin{aligned} P(w_1 \cap w_2) &= P(w_1)P(w_2|w_1) \\ &= \left(\tfrac{2}{5}\right)\left(\tfrac{1}{4}\right) = \tfrac{1}{10} \end{aligned}$$

The combined outcome $w_1 \cap w_2$ corresponds to the unique path Sw_1w_2 in the tree diagram, and we see that the probability of reaching w_2 along this path is just the product of the probabilities assigned to the branches on the path. Reasoning in the same way, we obtain the probability of each remaining combined outcome by multiplying the probabilities assigned to the branches on the path corresponding to the given combined outcomes. These probabilities are often written at the ends of the paths to which they correspond (Fig. 7).

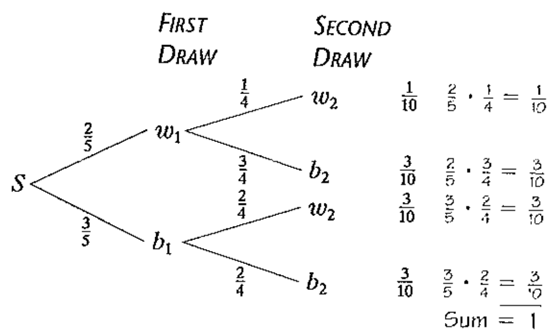


FIGURE 7

Now it is an easy matter to complete the problem. A white ball drawn on the second draw corresponds to either the combined outcome $w_1 \cap w_2$ or $b_1 \cap w_2$ occurring. Thus, since these combined outcomes are mutually exclusive,

$$\begin{aligned} P(w_2) &= P(w_1 \cap w_2) + P(b_1 \cap w_2) \\ &= \tfrac{1}{10} + \tfrac{3}{10} = \tfrac{4}{10} = \tfrac{2}{5} \end{aligned}$$

which is just the sum of the probabilities listed at the ends of the two paths terminating in w_2.

*The sample space for the combined outcomes is $S = \{w_1w_2, w_1b_2, b_1w_2, b_1b_2\}$. If we let $w_1 = \{w_1w_2, w_1b_2\}$ and $w_2 = \{w_1w_2, b_1w_2\}$, then $w_1 \cap w_2 = \{w_1w_2\}$.

Matched Problem 4 Two balls are drawn in succession without replacement from a box containing 4 red and 2 white balls. What is the probability of drawing a red ball on the second draw?

The sequence of two experiments in Example 4 is an example of a *stochastic process*. In general, a **stochastic process** involves a sequence of experiments where the outcome of each experiment is not certain. Our interest is in making predictions about the process as a whole. The analysis in Example 4 generalizes to stochastic processes involving any finite sequence of experiments. We summarize the procedures used in Example 4 for general application:

Constructing Probability Trees

Step 1. Draw a tree diagram corresponding to all combined outcomes of the sequence of experiments.

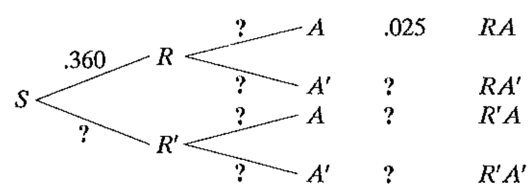
Step 2. Assign a probability to each tree branch. (This is the probability of the occurrence of the event on the right end of the branch subject to the occurrence of all events on the path leading to the event on the right end of the branch. The probability of the occurrence of a combined outcome that corresponds to a path through the tree is the product of all branch probabilities on the path.*)

Step 3. Use the results in steps 1 and 2 to answer various questions related to the sequence of experiments as a whole.

*If we form a sample space S such that each simple event in S corresponds to one path through the tree, and if the probability assigned to each simple event in S is the product of the branch probabilities on the corresponding path, then it can be shown that this is not only an acceptable assignment (all probabilities for the simple events in S are nonnegative and their sum is 1), but it is the only assignment consistent with the method used to assign branch probabilities within the tree.

Explore–Discuss 2

Refer to the table on rain and accidents in Example 2 and use formula (1), where appropriate, to complete the following probability tree:



Discuss the difference between $P(R \cap A)$ and $P(A|R)$.

Example 5 **Product Defects** A large computer company A subcontracts the manufacturing of its circuit boards to two companies, 40% to company B and 60% to company C. Company B in turn subcontracts 70% of the orders it receives from company A to company D and the remaining 30% to company E, both subsidiaries of company B. When the boards are completed by companies D, E, and C, they are shipped to company A to be used in various computer models. It has been found that 1.5%, 1%, and .5% of the boards from D, E, and C, respectively, prove defective during the 90-day warranty period after a computer is first sold. What is the probability that a given board in a computer will be defective during the 90-day warranty period?

SOLUTION Draw a tree diagram and assign probabilities to each branch (Fig. 8):

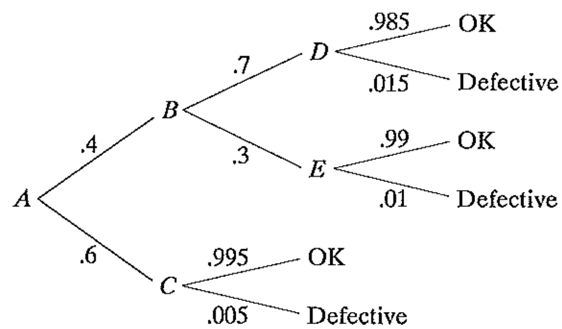


FIGURE 8

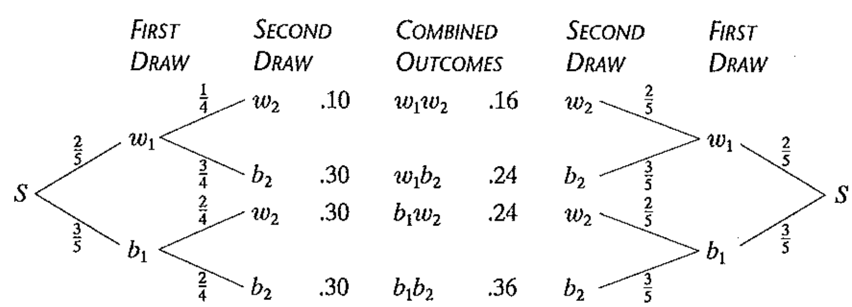
There are three paths leading to defective (the board will be defective within the 90-day warranty period). We multiply the branch probabilities on each path and add the three products:

$$\begin{aligned} P(\text{defective}) &= (.4)(.7)(.015) + (.4)(.3)(.01) + (.6)(.005) \\ &= .0084 \end{aligned}$$

Matched Problem 5 In Example 5, what is the probability that a circuit board in a completed computer came from company E or C?

Independent Events

We return to Example 4, which involved drawing two balls in succession without replacement from a box of 3 blue and 2 white balls (Fig. 4). What difference does "without replacement" and "with replacement" make? Figure 9 shows probability trees corresponding to each case. Go over the probability assignments for the branches in part (B) to convince yourself of their correctness.



(A) Without replacement (B) With replacement

FIGURE 9 $S = \{w_1w_2, w_1b_2, b_1w_2, b_1b_2\}$

Let: A = white ball on second draw $= \{w_1w_2, b_1w_2\}$
B = white ball on first draw $= \{w_1w_2, w_1b_2\}$

We now compute $P(A|B)$ and $P(A)$ for each case in Figure 9.

Case 1. *Without replacement:*

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P\{w_1w_2\}}{P\{w_1w_2, w_1b_2\}} = \frac{.10}{.10 + .30} = .25$$

(This is the assignment to branch w_1w_2 we made by looking in the box and counting.)

$$P(A) = P\{w_1w_2, b_1w_2\} = .10 + .30 = .40$$

Note that $P(A|B) \neq P(A)$, and we conclude that the probability of A is affected by the occurrence of B.

Case 2. *With replacement:*

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P\{w_1w_2\}}{P\{w_1w_2, w_1b_2\}} = \frac{.16}{.16 + .24} = .40$$

(This is the assignment to branch w_1w_2 we made by looking in the box and counting.)

$$P(A) = P\{w_1w_2, b_1w_2\} = .16 + .24 = .40$$

Note that $P(A|B) = P(A)$, and we conclude that the probability of A is not affected by the occurrence of B.

Intuitively, if $P(A|B) = P(A)$, then it appears that event A is "independent" of B. Let us pursue this further. If events A and B are such that

$$P(A|B) = P(A)$$

then replacing the left side by its equivalent from equation (1), we obtain

$$\frac{P(A \cap B)}{P(B)} = P(A)$$

After multiplying both sides by $P(B)$, the last equation becomes

$$P(A \cap B) = P(A)P(B)$$

This result motivates the following definition of *independence:*

Independence

If A and B are any events in a sample space S, we say that **A and B are independent** if and only if

$$P(A \cap B) = P(A)P(B) \tag{3}$$

Otherwise, A and B are said to be **dependent.**

From the definition of independence one can prove (see Problems 43 and 44, Exercise 6-5) the following theorem:

THEOREM 1 On Independence

If A and B are independent events with nonzero probabilities in a sample space S, then

$$P(A|B) = P(A) \qquad \text{and} \qquad P(B|A) = P(B) \tag{4}$$

If either equation in (4) holds, then A and B are independent.