Mathematics 19
Introduction to Set Theory
Spring 2022
Tentative Syllabus 3/16/2022

Instructor: M. Groszek
Distributive: QDS
Prerequisites: None

Around the beginning of the nineteenth century, efforts to place mathematics on an unassailable logical foundation collided with Russell's Paradox. This paradox was described by Russell in 1902.[1] It shows that the operation of set formation, gathering all objects sharing a given property into a set, can lead to fatal contradiction. Since set formation was being treated as intuitive and central to both logic and mathematics, this was an intellectual catastrophe. In fact, in 1931, Kurt Gödel's Incompleteness Theorems showed that a truly unassailable logical foundation for mathematics is impossible.[2]

Set theory, which had first revealed the problem, provided a solution. Georg Cantor began the development of modern set theory in the latter part of the eighteenth century. Cantor was aware of paradoxes in set theory (Russell's Paradox is really a variation of Cantor's Paradox), but was content to say that some collections of objects are "inconsistent multiplicities" that cannot be collected into a set.[3] In 1908 Ernst Zermelo published a collection of axioms for set theory,[4] which carefully enumerate what kinds of collections of objects can be formed into sets, and apparently avoid the paradoxes of Cantor, Russell, and others. ("Apparently" is the best we can do, as Gödel's Theorem means it is impossible to prove this.) The axiomatization of set theory allows us to place mathematics on a set-theoretical foundation that, if not entirely unassailable, has proved both defensible and lasting.

[1]Russell, B., 1902. "Letter to Frege," in Jean van Heijenoort (ed.), *From Frege to Gödel*, Cambridge, Mass.: Harvard University Press, 1967, 124-125.

[2]Gödel, K., 1931, "Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I," Monatshefte für Mathematik Physik, 38: 173-198. English translation in Jean van Heijenoort (ed.), *From Frege to Gödel*, Cambridge, Mass.: Harvard University Press, 1967, 596-616

[3]Cantor, G., 1899. "Letter to Dedekind," in jean van Heijenoort (ed.), *from Frege to Gödel*, Cambridge, Mass.: Harvard University Press, 1967, 124-125.

[4]Zermelo, E., 1908, "Üntersuchungen über die Grundlagen der Mengenlehre I," Mathematische Annalen 65: 261-281. English translation in Jean van Heijenoort (ed.), *From Frege to Gödel*, Cambridge, Mass.: Harvard University Press, 1967, 596-616

Set theory is not only a foundation for mathematics, it is a beautiful and important subject in its own right. Mathematician David Hilbert famously described it as "the paradise that Cantor created for us."[5]

This course takes the axiomatic approach advanced by Zermelo. We will learn the Zermelo-Fraenkel axioms for set theory and see how they can be justified. We will use these axioms to develop a highly structured picture of the set-theoretic universe. We will see how set theory becomes a foundation for mathematics by using sets to represent various kinds of mathematical objects. We will learn some techniques for proving things about sets, and we will begin to develop the skill of writing clear and logically correct mathematical arguments. Because set theory has been so widely adopted as a foundation for mathematics, pretty much all upper-level mathematics courses use some of the language and techniques of set theory, which makes Math 19 good preparation for further study of mathematics.

Learning objectives:

Students will be able to give reasons for taking an axiomatic approach to set theory.

Students will be able to use the language and basic notions of set theory in reading textbooks, solving problems, and writing proofs, in this and in subsequent courses.

Students will be able to write clear and correct proofs. This entails using, and communicating, valid logical reasoning.

Students will know the axioms of set theory, will be able to state them correctly (if informally), will be able to explain why they are justified based on the intuitive picture of the set theoretic universe being built up from below, and will be able to use them in solving problems and writing proofs. Writing proofs means that students will be able both to figure out how to prove things, at a level of difficulty appropriate to an introductory mathematics course, and to communicate their arguments.

Students will know the important theorems and definitions, will be able to state them correctly, and will be able to use them in solving problems and writing proofs.

[5]Hilbert, D., 1926, "Über das Unendliche", Mathematische Annalen, 95: 161-190. Lecture given Münster, 4 June 1925. English translation in Jean van Heijenoort (ed.), *From Frege to Gödel*, Cambridge, Mass.: Harvard University Press, 1967, 367-392.

Textbook: *Elements of Set Theory*, Herbert B. Enderton,
Academic Press, 1977. ISBN: 978-0122384400

Classes: 12 period (MWF 12:50-1:55, x-hour Tu 1:20-2:10), Kemeny 108.
We will meet during some x-hours for a proof-writing workshop.

Instructor: M. Groszek
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Office hours: TBA

Grading (tentative):
Homework: 30%
Exam 1: 15%
Exam 2: 25%
Exam 3: 30%

Homework: There will be three kinds of homework, all submitted on Canvas.

Reading homework will be due each class day, and will have a written component due the previous night, graded credit or no credit. If you do the assignment you will get credit.

Proof-writing homework will be due most Mondays, and will be graded credit or no credit. If you do not get credit, you will have a chance to redo the assignment for credit.

Regular homework will be due every Monday. Each problem will be graded for both correctness and clarity, on a scale of 0-5.

Late homework will get partial credit. (More details later.) The lowest three reading homework grades will be dropped.

There will be a homework policy document with more details.

Exams: There will be an exam in week 4 on the material covered in weeks 1-3, an exam in week 7 on the material covered in weeks 4-6, and a cumulative final exam concentrating on the material covered in weeks 7-9.

Each exam will have an in-class quiz portion and a take-home exam portion. The quiz portion will not be given during class time, but at a time to be determined. You will be able to take as much time as you need.

The Honor Principle :

Academic integrity is at the core of our mission as mathematicians and educators, and we take it very seriously. We also believe in working and learning together.

The Honor Principle (Homework):

Collaboration is permitted and encouraged, but no copying, and to be clear, this means no copying even from a board or scrap of paper on which a solution was hashed out collaboratively. What a student turns in as a homework solution is to be his or her own understanding of how to do the problems. The solutions you submit must be written by you alone. Any copying (electronic or otherwise) of another person's solutions, in whole or in part, is a violation of the Academic Honor Code.

Moreover, if in working with someone they have provided you with an important idea or approach, they should be explicitly given credit in your writeup. Hints given in office hours need not be cited. It is not necessarily sufficient to annotate your paper with a phrase like "I worked with Lee on all the problems." Individual ideas are to be credited at each instance; they represent intellectual property. Giving another person credit for an idea will never lower your grade. You get credit for your understanding no matter how you acquired it.

The Honor Principle (Exams):

On in-class quiz sections of exams, you may not receive assistance of any kind from any source (living, published, electronic, etc), except the professor, you may not consult any written materials including your notes and textbook, and you may not give assistance to anyone.

On take-home sections of exams, you may use your textbook, your own notes and homework, and any materials distributed in class, but you may not receive help from any other person, and you may not use outside sources, including but not limited to other textbooks and online sources. Matters of clarification are to be left to the professor. If you have any questions as to whether some action would be acceptable under the Academic Honor Code, please speak to me, and I will be glad to help clarify things. It is always easier to ask beforehand.

Disability Accommodations, Religious Observances, and Other Concerns

The following policies address some common concerns students may have, but certainly not all of them. If you have any questions or worries about the course or your participation in it, please talk to your instructor. We want all students to succeed.

• Students with disabilities who may need disability-related academic adjustments and services for this course are encouraged to see their instructor privately as early in the term as possible. Students requiring disability-related academic adjustments and services must consult the Student Accessibility Services office (Carson Hall, Suite 125, 646-9900, Student.Accessibility.Services@Dartmouth.edu).

Once SAS has authorized services, students must show the originally signed SAS Services and Consent Form and/or a letter on SAS letterhead to their instructor. As a first step, if you have questions about whether you qualify to receive academic adjustments and services, you should contact the SAS office. All inquiries and discussions will remain confidential.

• The academic environment at Dartmouth is challenging, our terms are intensive, and classes are not the only demanding part of your life. There are a number of resources available to you on campus to support your wellness, including your undergraduate dean, Counseling and Human Development, and the Student Wellness Center.

• The Sexual Respect Website (https://sexual-respect.dartmouth.edu) at Dartmouth provides a wealth of information on your rights with regard to sexual respect and resources that are available to all in our community.

Please note that, as a faculty member, I am obligated to share disclosures regarding conduct under Title IX with Dartmouth's Title IX Coordinator. Confidential resources are also available, and include licensed medical or counseling professionals (e.g., a licensed psychologist), staff members of organizations recognized as rape crisis centers under state law (such as WISE), and ordained clergy(see https://dartgo.org/titleix_resources) .

Should you have any questions, please feel free to contact Dartmouth's Title IX Coordinator or the Deputy Title IX Coordinator for the Guarini School. Their contact information can be found on the sexual respect websiteat: https://sexual-respect.dartmouth.edu .

• Some students may wish to take part in religious observances that occur during this academic term. If you have a religious observance that conflicts with your participation in the course, please meet with your instructor before the end of the second week of the term to discuss appropriate accommodations.

• It may be possible to accommodate conflicts with extracurricular activities, employment, or family responsibilities, although this is not guaranteed. Please meet with your instructor as soon as possible.

This class is planned to be in-person. If circumstances require some classes be held remotely, the following policies are particularly relevant.

Classes, privacy, and intellectual property:

By enrolling in this course, you accept and agree to the College's Consent to Record Statement that follows this short discussion.

Only the instructor may record any Math 19 meetings, including class sessions, breakout rooms, x-hours, and office hours.

I do not intend to record any classes. If I do make any recordings, it will be announced at the time. I will not record office hours.

Please do not share any recordings beyond our class, and do not invite anyone else to observe our Zoom class sessions. Students participating in class should expect that class proceedings are not made public beyond those with legitimate access.

Consent to Record Statement:

(1) Consent to recording of course meetings and office hours that are open to multiple students.

By enrolling in this course,

a) I affirm my understanding that the instructor may record meetings of this course and any associated meetings open to multiple students and the instructor, including but not limited to scheduled and ad hoc office hours and other consultations, within any digital platform, including those used to offer remote instruction for this course.

b) I further affirm that the instructor owns the copyright to their instructional materials, of which these recordings constitute a part, and my distribution of any of these recordings in whole or in part to any person or entity other than other members of the class without prior written consent of the instructor may be subject to discipline by Dartmouth up to and including separation from Dartmouth.

(2) Requirement of consent to one-on-one recordings

By enrolling in this course, I hereby affirm that I will not make a recording in any medium of any one-on-one meeting with the instructor or another member of the class or group of members of the class without obtaining the prior written consent of all those participating, and I understand that if I violate this prohibition, I will be subject to discipline by Dartmouth up to and including separation from Dartmouth, as well as any other civil or criminal penalties under applicable law. I understand that an exception

to this consent applies to accommodations approved by SAS for a student's disability, and that one or more students in a class may record class lectures, discussions, lab sessions, and review sessions and take pictures of essential information, and/or be provided class notes for personal study use only.

If you have questions, please contact the Office of the Dean of the Faculty of Arts and Sciences.

Math 19 Spring 2022
Tentative Schedule

1. March 28-April 1

   (a) Monday: Introduction to the course.
   Chapter 1, section (a), pages 1-7.

   (b) Tuesday (x-hour): Proof-writing workshop.

   (c) Wednesday: Overview of axiomatic set theory.
   Chapter 1, sections (b)-(f), pages 7-16.

   (d) Friday: The first axioms.
   Chapter 2, sections (a)-(b), pages 17-26.

2. April 4-8

   (a) Monday: Algebra of sets.
   Chapter 2, sections (c)-(e), pages 27-34.

   (b) Tuesday (x-hour): Proof-writing workshop.

   (c) Wednesday: Ordered pairs and relations.
   Chapter 3, sections (a)-(c), pages 35-42.

   (d) Friday: Functions.
   Chapter 3, sections (d)-(e), pages 42-55.

3. April 11-15

   (a) Monday: Equivalence relations.
   Chapter 3, section (f), pages 55-62.

   (b) Tuesday (x-hour): Proof-writing workshop.

   (c) Wednesday: Ordering relations.
   Chapter 3, sections (g)-(h), pages 62-66.

   (d) Friday: Natural numbers.
   Chapter 4, sections (a)-(b), pages 67-73.

4. April 18-22

   Exam this week on material covered in weeks 1-3.

(a) Monday: Recursion on ω.
Chapter 4, section (c), pages 73-78.

(b) Tuesday (x-hour): No x-hour this week.

(c) Wednesday: Arithmetic.
Chapter 4, section (d), pages 79-82.

(d) Friday: Ordering the natural numbers.
Chapter 4, sections (e)-(f), pages 83-89.

5. April 25-29

(a) Monday: Integers.
Chapter 5, section (a), pages 90-101.

(b) Tuesday (x-hour): Proof-writing workshop.

(c) Wednesday: Representing mathematical objects as sets.
Chapter 5, section (e), pages 123-127.

(d) Friday: Sizes of sets.
Chapter 6, section (a), pages 128-133.

6. May 2-6

(a) Monday: Finite sets.
Chapter 6, Section (b), pages 133-138.

(b) Tuesday (x-hour): Proof-writing workshop.

(c) Wednesday: Cardinal arithmetic.
Chapter 6, section (c), pages 138-145.

(d) Friday: Ordering cardinal numbers.
Chapter 6, section (d), pages 145-151.

7. May 9-13

Exam this week on material in weeks 1-6 (mostly weeks 4-6).

(a) Monday: The axiom of choice.
Chapter 6, section (e), pages 151-159.

(b) Tuesday (x-hour): No x-hour today.

(c) Wednesday: Infinite cardinals.
Chapter 6 sections (f)-(h), pages 159-166.

(d) Friday: Partial orderings.
Chapter 7, section (a), pages 167-172.

8. May 16-20

(a) Monday: Well orderings.
Chapter 7, section (b), pages 172-179.

(b) Tuesday (x-hour): Proof-writing workshop.

(c) Wednesday: Replacement axioms.
Chapter 7, section (c), pages 179-18

(d) Friday: Epsilon images and isomorphisms.
Chapter 7, sections (d)-(e), pages 182-187.

9. May 23-27

(a) Monday: Ordinal numbers.
Chapter 7, section (f), pages 187-195.

(b) Tuesday (x-hour): Proof-writing workshop.

(c) Wednesday: Cardinal numbers.
Chapter 7, section (g), pages 195-200.

(d) Friday: Rank.
Chapter 7, section (h), pages 200-208.

10. May 30-June 3

(a) Monday: No class; Memorial Day.

(b) Tuesday (x-hour): No x-hour today.

(c) Wednesday: Conclusion.
Last day of class.