

1.2 Counting Lists, Permutations, and Subsets.

Using the Sum and Product Principles

Exercise 1.2-1 A password for a certain computer system is supposed to be between 4 and 8 characters long and composed of lower and/or upper case letters. How many passwords are possible? What counting principles did you use? Estimate the percentage of the possible passwords that have exactly four characters.

A good way to attack a counting problem is to ask if we could use either the sum principle or the product principle to simplify or completely solve it. Here that question might lead us to think about the fact that a password can have 4, 5, 6, 7 or 8 characters. The set of all passwords is the union of those with 4, 5, 6, 7, and 8 letters so the sum principle might help us. To write the problem algebraically, let P_i be the set of i -letter passwords and P be the set of all possible passwords. Clearly,

$$P = P_4 \cup P_5 \cup P_6 \cup P_7 \cup P_8 .$$

The P_i are mutually disjoint, and thus we can apply the sum principle to obtain

$$|P| = \sum_{i=4}^8 |P_i| .$$

We still need to compute $|P_i|$. For an i -letter password, there are 52 choices for the first letter, 52 choices for the second and so on. Thus by the product principle, $|P_i|$, the number of passwords with i letters is 52^i . Therefore the total number of passwords is

$$52^4 + 52^5 + 52^6 + 52^7 + 52^8 .$$

Of these, 52^4 have four letters, so the percentage with 54 letters is

$$\frac{100 \cdot 52^4}{52^4 + 52^5 + 52^6 + 52^7 + 52^8} .$$

Although this is a nasty formula to evaluate by hand, we can get a quite good estimate as follows. Notice that 52^8 is 52 times as big as 52^7 , and even more dramatically larger than any other term in the sum in the denominator. Thus the ratio thus just a bit less than

$$\frac{100 \cdot 52^4}{52^8} ,$$

which is $100/52^4$, or approximately .000014. Thus to five decimal places, only .00001% of the passwords have four letters. It is therefore much easier guess a password that we know has four letters than it is to guess one that has between 4 and 8 letters—roughly 7 million times easier!

In our solution to Exercise 1.2-1, we casually referred to the use of the product principle in computing the number of passwords with i letters. We didn't write any set as a union of sets of equal size. We could have, but it would have been clumsy and repetitive. For this reason we will state a second version of the product principle that we can derive from the version for unions of sets by using the idea of mathematical induction that we study in Chapter 4.

Version 2 of the *product principle* states:

Principle 1.4 (Product Principle, Version 2) *If a set S of lists of length m has the properties that*

1. *There are i_1 different first elements of lists in S , and*
2. *For each $j > 1$ and each choice of the first $j - 1$ elements of a list in S there are i_j choices of elements in position j of those lists,*

then there are $i_1 i_2 \cdots i_m$ lists in S .

Let's apply this version of the product principle to compute the number of m -letter passwords. Since an m -letter password is just a list of m letters, and since there are 52 different first elements of the password and 52 choices for each other position of the password, we have that $i_1 = 52$, $i_2 = 52, \dots, i_m = 52$. Thus, this version of the product principle tells us immediately that the number of passwords of length m is $i_1 i_2 \cdots i_m = 52^m$.

Lists and functions

We have left a term undefined in our discussion of version 2 of the product principle, namely the word list. A list of 3 things chosen from a set T consists of a first member t_1 of T , a second member t_2 of T , and a third member t_3 of T . If we rewrite the list in a different order, we get a different list. A list of k things chosen from T consists of a first member of T through a k th member of T . We can use the word function, which you probably recall from algebra or calculus, to be more precise.

Recall that a function from a set S (called the *domain* of the function) to a set T (called the *range* of the function) is a relationship between the elements of S and the elements of T that relates exactly one element of T to each element of S . We use a letter like f to stand for a function and use $f(x)$ to stand for the one and only one element of T that the function relates to the element x of S . You are probably used to thinking of functions in terms of formulas like $f(x) = x^2$. We need to use formulas like this in algebra and calculus because the functions that you study in algebra and calculus have infinite sets of numbers as their domains and ranges. In discrete mathematics, functions often have finite sets as their domains and ranges, and so it is possible to describe a function by saying exactly what it is. For example

$$f(1) = \text{Sam}, f(2) = \text{Mary}, f(3) = \text{Sarah}$$

is a function that describes a list of three people. This suggests a precise definition of a list of k elements from a set T : A *list of k elements* from a set T is a function from $\{1, 2, \dots, k\}$ to T .

Exercise 1.2-2 Write down all the functions from the two-element set $\{1, 2\}$ to the two-element set $\{a, b\}$.

Exercise 1.2-3 How many functions are there from a two-element set to a three element set?

Exercise 1.2-4 How many functions are there from a three-element set to a two-element set?

In Exercise 1.2-2 one thing that is difficult is to choose a notation for writing the functions down. We will use f_1, f_2 , etc., to stand for the various functions we find. To describe a function f_i from $\{1, 2\}$ to $\{a, b\}$ we have to specify $f_i(1)$ and $f_i(2)$. We can write

$$\begin{aligned} f_1(1) &= a & f_1(2) &= b \\ f_2(1) &= b & f_2(2) &= a \\ f_3(1) &= a & f_3(2) &= a \\ f_4(1) &= b & f_4(2) &= b \end{aligned}$$

We have simply written down the functions as they occurred to us. How do we know we have all of them? The set of all functions from $\{1, 2\}$ to $\{a, b\}$ is the union of the functions f_i that have $f_i(1) = a$ and those that have $f_i(1) = b$. The set of functions with $f_i(1) = a$ has two elements, one for each choice of $f_i(2)$. Therefore by the product principle the set of all functions from $\{1, 2\}$ to $\{a, b\}$ has size $2 \cdot 2 = 4$.

To compute the number of functions from a two element set (say $\{1, 2\}$) to a three element set, we can again think of using f_i to stand for a typical function. Then the set of all functions is the union of three sets, one for each choice of $f_i(1)$. Each of these sets has three elements, one for each choice of $f_i(2)$. Thus by the product principle we have $3 \cdot 3 = 9$ functions from a two element set to a three element set.

To compute the number of functions from a three element set (say $\{1, 2, 3\}$) to a two element set, we observe that the set of functions is a union of four sets, one for each choice of $f_i(1)$ and $f_i(2)$ (as we saw in our solution to Exercise 1.2-2). But each of these sets has two functions in it, one for each choice of $f_i(3)$. Then by the product principle, we have $4 \cdot 2 = 8$ functions from a three element set to a two element set.

A function f is called *one-to-one* or an *injection* if whenever $x \neq y$, $f(x) \neq f(y)$. Notice that the two functions f_1 and f_2 we gave in our solution of Exercise 1.2-2 are one-to-one, but f_3 and f_4 are not.

A function f is called *onto* or a *surjection* if every element y in the range is $f(x)$ for some x in the domain. Notice that the functions f_1 and f_2 in our solution of Exercise 1.2-2 are onto functions but f_3 and f_4 are not.

Exercise 1.2-5 Using two-element sets or three-element sets as domains and ranges, find an example of a one-to-one function that is not onto.

Exercise 1.2-6 Using two-element sets or three-element sets as domains and ranges, find an example of an onto function that is not one-to-one.

Notice that the function given by $f(1) = c, f(2) = a$ is an example of a function from $\{1, 2\}$ to $\{a, b, c\}$ that is one-to-one but not onto.

Notice that the function given by $f(1) = a, f(2) = b, f(3) = a$ is an example of a function from $\{1, 2, 3\}$ to $\{a, b\}$ that is onto but not one-to-one.

The Bijection Principle

Exercise 1.2-7 The loop below is part of a program to determine the number of triangles formed by n points in the plane.

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(1) trianglecount = 0
(2) for i = 1 to n
(3)     for j = i + 1 to n
(4)         for k = j + 1 to n
(5)             if points i, j, and k are not collinear
(6)                 trianglecount = trianglecount + 1

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How many times does the above code check three points to see if they are collinear in line 5?

In Exercise 1.2-7, we have a loop embedded in a loop that is embedded in another loop. Because the second loop, starting in line 3, begins with $j = i + 1$ and j increase up to n , and because the third loop, starting in line 4, begins with $k = j + 1$ and increases up to n , our code examines each triple of values i, j, k with $i < j < k$ exactly once. For example, if n is 4, then the triples (i, j, k) used by the algorithm, in order, are $(1, 2, 3)$, $(1, 2, 4)$, $(1, 3, 4)$, and $(2, 3, 4)$. Thus one way in which we might have solved Exercise 1.2-7 would be to compute the number of such triples, which we will call *increasing triples*. As with the case of two-element subsets earlier, the number of such triples is the number of three-element subsets of an n -element set. This is the second time that we have proposed counting the elements of one set (in this case the set of increasing triples chosen from an n -element set) by saying that it is equal to the number of elements of some other set (in this case the set of three element subsets of an n -element set). When are we justified in making such an assertion that two sets have the same size? There is another fundamental principle that abstracts our concept of what it means for two sets to have the same size. Intuitively two sets have the same size if we can match up their elements in such a way that each element of one set corresponds to exactly one element of the other set. This description carries with it some of the same words that appeared in the definitions of functions, one-to-one, and onto. Thus it should be no surprise that one-to-one and onto functions are part of our abstract principle.

Principle 1.5 (Bijection Principle) *Two sets have the same size if and only if there is a one-to-one function from one set onto the other.*

Our principle is called the *bijection principle* because a one-to-one and onto function is called a *bijection*. Another name for a bijection is a *one-to-one correspondence*. A bijection from a set to itself is called a *permutation* of that set.

What is the bijection that is behind our assertion that the number of increasing triples equals the number of three-element subsets? We define the function f to be the one that takes the increasing triple (i, j, k) to the subset $\{i, j, k\}$. Since the three elements of an increasing triple are different, the subset is a three element set, so we have a function from increasing triples to three element sets. Two different triples can't be the same set in two different orders, so different triples have to be associated with different sets. Thus f is one-to-one. Each set of three integers can be listed in increasing order, so it is the image under f of an increasing triple. Therefore f is onto. Thus we have a one-to-one correspondence, or bijection, between the set of increasing triples and the set of three element sets.

k -element permutations of a set

Since counting increasing triples is equivalent to counting three-element subsets, we can count increasing triples by counting three-element subsets instead. We use a method similar to the one we used to compute the number of two-element subsets of a set. Recall that the first step was to compute the number of ordered pairs of distinct elements we could choose from the set $\{1, 2, \dots, n\}$. So we now ask in how many ways may we choose an ordered triple of distinct elements from $\{1, 2, \dots, n\}$, or more generally, in how many ways may we choose a list of k distinct elements from $\{1, 2, \dots, n\}$. A list of k -distinct elements chosen from a set N is called a k -element permutation of N .⁴

How many 3-element permutations of $\{1, 2, \dots, n\}$ can we make? Recall that a k -element permutation is a list of k distinct elements. There are n choices for the first number in the list. For each way of choosing the first element, there are $n - 1$ choices for the second. For each choice of the first two elements, there are $n - 2$ ways to choose a third (distinct) number, so by version 2 of the product principle, there are $n(n - 1)(n - 2)$ ways to choose the list of numbers. For example, if n is 4, the three-element permutations of $\{1, 2, 3, 4\}$ are

$$L = \{123, 124, 132, 134, 142, 143, 213, 214, 231, 234, 241, 243, \\ 312, 314, 321, 324, 341, 342, 412, 413, 421, 423, 431, 432\}. \quad (1.4)$$

There are indeed $4 \cdot 3 \cdot 2 = 24$ lists in this set. Notice that we have listed the lists in the order that they would appear in a dictionary (assuming we treated numbers as we treat letters). This ordering of lists is called the *lexicographic ordering*.

A general pattern is emerging. To compute the number of k -element permutations of the set $\{1, 2, \dots, n\}$, we recall that they are lists and note that we have n choices for the first element of the list, and regardless of which choice we make, we have $n - 1$ choices for the second element of the list, and more generally, given the first $i - 1$ elements of a list we have $n - (i - 1) = n - i + 1$ choices for the i th element of the list. Thus by version 2 of the product principle, we have $n(n - 1) \cdots (n - k + 1)$, which is the first k terms of $n!$, ways to choose a k -element permutation of $\{1, 2, \dots, n\}$. There is a very handy notation for this product first suggested by Don Knuth. We use $n^{\underline{k}}$ to stand for $n(n - 1) \cdots (n - k + 1)$, and call it the k th falling factorial power of n . We can summarize our observations in a theorem.

Theorem 1.1 *The number k -element permutations of an n -element set is*

$$n^{\underline{k}} = n(n - 1) \cdots (n - k + 1) = n! / (n - k)! .$$

Counting subsets of a set

We now return to the question of counting the number of three element subsets of a $\{1, 2, \dots, n\}$. We use $\binom{n}{3}$, which we read as “ n choose 3” to stand for the number of three element subsets of

⁴In particular a k -element permutation of $\{1, 2, \dots, k\}$ is a list of k distinct elements of $\{1, 2, \dots, k\}$, which, by our definition of a list is a function from $\{1, 2, \dots, k\}$ to $\{1, 2, \dots, k\}$. This function must be one-to-one since the elements of the list are distinct. Since there are k distinct elements of the list, every element of $\{1, 2, \dots, k\}$ appears in the list, so the function is onto. Therefore it is a bijection. Thus our definition of a permutation of a set is consistent with our definition of a k -element permutation in the case where the set is $\{1, 2, \dots, k\}$.

$\{1, 2, \dots, n\}$, or more generally of any n -element set. We have just carried out the first step of computing $\binom{n}{3}$ by counting the number of three-element permutations of $\{1, 2, \dots, n\}$.

Exercise 1.2-8 Let L be the set of all three-element permutations of $\{1, 2, 3, 4\}$, as in Equation 1.4. How many of the lists (permutations) in L are lists of the 3 element set $\{1, 3, 4\}$? What are these lists?

We see that this set appears in L as 6 different lists: 134, 143, 314, 341, 413, and 431. In general given three different numbers with which to create a list, there are three ways to choose the first number in the list, given the first there are two ways to choose the second, and given the first two there is only one way to choose the third element of the list. Thus by version 2 of the product principle once again, there are $3 \cdot 2 \cdot 1 = 6$ ways to make the list.

Since there are $n(n-1)(n-2)$ permutations of an n -element set, and each three-element subset appears in exactly 6 of these lists, the number of three-element permutations is six times the number of three element subsets. That is, $n(n-1)(n-2) = \binom{n}{3} \cdot 6$. Whenever we see that one number that counts something is the product of two other numbers that count something, we should expect that there is an argument using the product principle that explains why. Thus we should be able to see how to break the set of all 3-element permutations of $\{1, 2, \dots, n\}$ into either 6 disjoint sets of size $\binom{n}{3}$ or into $\binom{n}{3}$ subsets of size six. Since we argued that each three element subset corresponds to six lists, we have described how to get a set of six lists from one three-element set. Two different subsets could never give us the same lists, so our sets of three-element lists are disjoint. In other words, we have divided the set of all three-element permutations into $\binom{n}{3}$ mutually sets of size six. In this way the product principle does explain why $n(n-1)(n-2) = \binom{n}{3} \cdot 6$. By division we get that we have

$$\binom{n}{3} = n(n-1)(n-2)/6$$

three-element subsets of $\{1, 2, \dots, n\}$. For $n = 4$, the number is $4(3)(2)/6 = 4$. These sets are $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$, and $\{2, 3, 4\}$. It is straightforward to verify that each of these sets appears 6 times in L , as 6 different lists.

Essentially the same argument gives us the number of k -element subsets of $\{1, 2, \dots, n\}$. We denote this number by $\binom{n}{k}$, and read it as “ n choose k .” Here is the argument: the set of all k -element permutations of $\{1, 2, \dots, n\}$ can be partitioned into $\binom{n}{k}$ disjoint blocks⁵, each block consisting of all k -element permutations of a k -element subset of $\{1, 2, \dots, n\}$. But the number of k -element permutations of a k -element set is $k!$, either by version 2 of the product principle or by Theorem 1.1. Thus by version 1 of the product principle we get the equation

$$n^k = \binom{n}{k} k!.$$

Division by $k!$ gives us our next theorem.

Theorem 1.2 For integers n and k with $0 \leq k \leq n$, the number of k element subsets of an n element set is

$$\frac{n^k}{k!} = \frac{n!}{k!(n-k)!}$$

⁵using the language introduced for partitions of sets in Section 1.1

Proof: The proof is given above, except in the case that k is 0; however the only subset of our n -element set of size zero is the empty set, so we have exactly one such subset. This is exactly what the formula gives us as well. (Note that the cases $k = 0$ and $k = n$ both use the fact that $0! = 1$.⁶) The equality in the theorem comes from the definition of n^k ■

Another notation for the numbers $\binom{n}{k}$ is $C(n, k)$. Thus we have that

$$C(n, k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}. \quad (1.5)$$

These numbers are called *binomial coefficients* for reasons that will become clear later.

Important Concepts, Formulas, and Theorems

1. *List.* A list of k items chosen from a set X is a function from $\{1, 2, \dots, k\}$ to X .
2. *Lists versus sets.* In a list, the order in which elements appear in the list matters, and an element may appear more than once. In a set, the order in which we write down the elements of the set does not matter, and an element can appear at most once.
3. *Product Principle, Version 2.* If a set S of lists of length m has the properties that
 - (a) There are i_1 different first elements of lists in S , and
 - (b) For each $j > 1$ and each choice of the first $j - 1$ elements of a list in S there are i_j choices of elements in position j of those lists,
 then there are $i_1 i_2 \cdots i_m$ lists in S .
4. *Function.* A function f from a set S to a set T is a relationship between S and T that relates exactly one element of T to each element of S . We write $f(x)$ for the one and only one element of T that the function f relates to the element x of S . The same element of T may be related to different members of S .
5. *Onto, Surjection* A function f from a set S to a set T is *onto* if for each element $y \in T$, there is at least one $x \in S$ such that $f(x) = y$. An onto function is also called a *surjection*.
6. *One-to-one, Injection.* A function f from a set S to a set T is *one-to-one* if, for each $x \in S$ and $y \in S$ with $x \neq y$, $f(x) \neq f(y)$. A one-to-one function is also called an *injection*.
7. *Bijection, One-to-one correspondence.* A function from a set S to a set T is a *bijection* if it is both one-to-one and onto. A bijection is sometimes called a *one-to-one correspondence*.
8. *Permutation.* A one-to-one function from a set S to S is called a *permutation* of S .
9. *k-element permutation.* A *k-element permutation* of a set S is a list of k distinct elements of S .
10. *k-element subsets. n choose k. Binomial Coefficients.* For integers n and k with $0 \leq k \leq n$, the number of k element subsets of an n element set is $n!/k!(n-k)!$. The number of k -element subsets of an n -element set is usually denoted by $\binom{n}{k}$ or $C(n, k)$, both of which are read as “ n choose k .” These numbers are called *binomial coefficients*.

⁶There are many reasons why $0!$ is defined to be one; making the formula for $\binom{n}{k}$ work out is one of them.

11. The number of k -element permutations of an n -element set is

$$n^{\underline{k}} = n(n-1) \cdots (n-k+1) = n!/(n-k)!$$

12. When we have a formula to count something and the formula expresses the result as a product, it is useful to try to understand whether and how we could use the product principle to prove the formula.

Problems

- The “Pile High Deli” offers a “simple sandwich” consisting of your choice of one of five different kinds of bread with your choice of butter or mayonnaise or no spread, one of three different kinds of meat, and one of three different kinds of cheese, with the meat and cheese “piled high” on the bread. In how many ways may you choose a simple sandwich?
- In how many ways can we pass out k distinct pieces of fruit to n children (with no restriction on how many pieces of fruit a child may get)?
- Write down all the functions from the three-element set $\{1, 2, 3\}$ to the set $\{a, b\}$. Indicate which functions, if any, are one-to-one. Indicate which functions, if any, are onto.
- Write down all the functions from the two element set $\{1, 2\}$ to the three element set $\{a, b, c\}$. Indicate which functions, if any, are one-to-one. Indicate which functions, if any, are onto.
- There are more functions from the real numbers to the real numbers than most of us can imagine. However in discrete mathematics we often work with functions from a finite set S with s elements to a finite set T with t elements. Then there are only a finite number of functions from S to T . How many functions are there from S to T in this case?
- Assuming $k \leq n$, in how many ways can we pass out k distinct pieces of fruit to n children if each child may get at most one? What is the number if $k > n$? Assume for both questions that we pass out all the fruit.
- Assume $k \leq n$, in how many ways can we pass out k identical pieces of fruit to n children if each child may get at most one? What is the number if $k > n$? Assume for both questions that we pass out all the fruit.
- What is the number of five digit (base ten) numbers? What is the number of five digit numbers that have no two consecutive digits equal? What is the number that have at least one pair of consecutive digits equal?
- We are making a list of participants in a panel discussion on allowing alcohol on campus. They will be sitting behind a table in the order in which we list them. There will be four administrators and four students. In how many ways may we list them if the administrators must sit together in a group and the students must sit together in a group? In how many ways may we list them if we must alternate students and administrators?
- (This problem is for students who are working on the relationship between k -element permutations and k -element subsets.) Write down all three element permutations of the five element set $\{1, 2, 3, 4, 5\}$ in lexicographic order. Underline those that correspond to the set $\{1, 3, 5\}$. Draw a rectangle around those that correspond to the set $\{2, 4, 5\}$. How many

three-element permutations of $\{1, 2, 3, 4, 5\}$ correspond to a given 3-element set? How many three-element subsets does the set $\{1, 2, 3, 4, 5\}$ have?

11. In how many ways may a class of twenty students choose a group of three students from among themselves to go to the professor and explain that the three-hour labs are actually taking ten hours?
12. We are choosing participants for a panel discussion allowing on allowing alcohol on campus. We have to choose four administrators from a group of ten administrators and four students from a group of twenty students. In how many ways may we do this?
13. We are making a list of participants in a panel discussion on allowing alcohol on campus. They will be sitting behind a table in the order in which we list them. There will be four administrators chosen from a group of ten administrators and four students chosen from a group of twenty students. In how many ways may we choose and list them if the administrators must sit together in a group and the students must sit together in a group? In how many ways may we choose and list them if we must alternate students and administrators?
14. In the local ice cream shop, you may get a sundae with two scoops of ice cream from 10 flavors (in accordance with your mother's rules from Problem 12 in Section 1.1, the way the scoops sit in the dish does not matter), any one of three flavors of topping, and any (or all or none) of whipped cream, nuts and a cherry. How many different sundaes are possible?
15. In the local ice cream shop, you may get a three-way sundae with three of the ten flavors of ice cream, any one of three flavors of topping, and any (or all or none) of whipped cream, nuts and a cherry. How many different sundaes are possible(in accordance with your mother's rules from Problem 12 in Section 1.1, the way the scoops sit in the dish does not matter) ?
16. A tennis club has $2n$ members. We want to pair up the members by twos for singles matches. In how many ways may we pair up all the members of the club? Suppose that in addition to specifying who plays whom, for each pairing we say who serves first. Now in how many ways may we specify our pairs?
17. A basketball team has 12 players. However, only five players play at any given time during a game. In how many ways may the coach choose the five players? To be more realistic, the five players playing a game normally consist of two guards, two forwards, and one center. If there are five guards, four forwards, and three centers on the team, in how many ways can the coach choose two guards, two forwards, and one center? What if one of the centers is equally skilled at playing forward?
18. Explain why a function from an n -element set to an n -element set is one-to-one if and only if it is onto.
19. The function g is called an *inverse* to the function f if the domain of g is the range of f , if $g(f(x)) = x$ for every x in the domain of f and if $f(g(y)) = y$ for each y in the range of f .
 - (a) Explain why a function is a bijection if and only if it has an inverse function.
 - (b) Explain why a function that has an inverse function has only one inverse function.