

### 1.3 Binomial Coefficients

In this section, we will explore various properties of binomial coefficients.

#### Pascal's Triangle

Table 1 contains the values of the binomial coefficients  $\binom{n}{k}$  for  $n = 0$  to 6 and all relevant  $k$  values. The table begins with a 1 for  $n = 0$  and  $k = 0$ , because the empty set, the set with no elements, has exactly one 0-element subset, namely itself. We have not put any value into the table for a value of  $k$  larger than  $n$ , because we haven't defined what we mean by the binomial coefficient  $\binom{n}{k}$  in that case. However, since there are no subsets of an  $n$ -element set that have size larger than  $n$ , it is natural to define  $\binom{n}{k}$  to be zero when  $k > n$ , and so we define  $\binom{n}{k}$  to be zero when  $k > n$ . Thus we could fill in the empty places in the table with zeros. The table is easier to read if we don't fill in the empty spaces, so we just remember that they are zero.

Table 1.1: A table of binomial coefficients

$n \setminus k$	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

**Exercise 1.3-1** What general properties of binomial coefficients do you see in Table 1.1

**Exercise 1.3-2** What is the next row of the table of binomial coefficients?

Several properties of binomial coefficients are apparent in Table 1.1. Each row begins with a 1, because  $\binom{n}{0}$  is always 1, as it must be because there is just one subset of an  $n$ -element set with 0 elements, namely the empty set. Similarly, each row ends with a 1, because an  $n$ -element set  $S$  has just one  $n$ -element subset, namely  $S$  itself. Each row increases at first, and then decreases. Further the second half of each row is the reverse of the first half. The array of numbers called *Pascal's Triangle* emphasizes that symmetry by rearranging the rows of the table so that they line up at their centers. We show this array in Table 2. When we write down Pascal's triangle, we leave out the values of  $n$  and  $k$ .

You may know a method for creating Pascal's triangle that does not involve computing binomial coefficients, but rather creates each row from the row above. Each entry in Table 1.2, except for the ones, is the sum of the entry directly above it to the left and the entry directly above it to the right. We call this the *Pascal Relationship*, and it gives another way to compute binomial coefficients without doing the multiplying and dividing in equation 1.5. If we wish to compute many binomial coefficients, the Pascal relationship often yields a more efficient way to do so. Once the coefficients in a row have been computed, the coefficients in the next row can be computed using only one addition per entry.

Table 1.2: Pascal's Triangle

				1				
				1	1			
			1	2	1			
		1	3	3	1			
	1	4	6	4	1			
	1	5	10	10	5	1		
1	6	15	20	15	6	1		

We now verify that the two methods for computing Pascal's triangle always yield the same result. In order to do so, we need an algebraic statement of the Pascal Relationship. In Table 1.1, each entry is the sum of the one above it and the one above it and to the left. In algebraic terms, then, the Pascal Relationship says

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}, \quad (1.6)$$

whenever  $n > 0$  and  $0 < k < n$ . Notice that It is possible to give a purely algebraic (and rather dreary) proof of this formula by plugging in our earlier formula for binomial coefficients into all three terms and verifying that we get an equality. A guiding principle of discrete mathematics is that when we have a formula that relates the numbers of elements of several sets, we should find an explanation that involves a relationship among the sets.

### A proof using sets

From Theorem 1.2 and Equation 1.5, we know that the expression  $\binom{n}{k}$  is the number of  $k$ -element subsets of an  $n$  element set. Each of the three terms in Equation 1.6 therefore represents the number of subsets of a particular size chosen from an appropriately sized set. In particular, the three sets are the set of  $k$ -element subsets of an  $n$ -element set, the set of  $(k-1)$ -element subsets of an  $(n-1)$ -element set, and the set of  $k$ -element subsets of an  $(n-1)$ -element set. We should, therefore, be able to explain the relationship between these three quantities using the sum principle. This explanation will provide a proof, just as valid a proof as an algebraic derivation. Often, a proof using subsets will be less tedious, and will yield more insight into the problem at hand.

Before giving such a proof in Theorem 1.3 below, we give an example. Suppose  $n = 5$ ,  $k = 2$ . Equation 1.6 says that

$$\binom{5}{2} = \binom{4}{1} + \binom{4}{2}. \quad (1.7)$$

Because the numbers are small, it is simple to verify this by using the formula for binomial coefficients, but let us instead consider subsets of a 5-element set. Equation 1.7 says that the number of 2 element subsets of a 5 element set is equal to the number of 1 element subsets of a 4 element set plus the number of 2 element subsets of a 4 element set. But to apply the sum principle, we would need to say something stronger. To apply the sum principle, we should be able to partition the set of 2 element subsets of a 5 element set into 2 disjoint sets, one of which

has the same size as the number of 1 element subsets of a 4 element set and one of which has the same size as the number of 2 element subsets of a 4 element set. Such a partition provides a proof of Equation 1.7. Consider now the set  $S = \{A, B, C, D, E\}$ . The set of two element subsets is

$$S_1 = \{\{A, B\}, \{AC\}, \{A, D\}, \{A, E\}, \{B, C\}, \{B, D\}, \{B, E\}, \{C, D\}, \{C, E\}, \{D, E\}\}.$$

We now partition  $S_1$  into 2 blocks,  $S_2$  and  $S_3$ .  $S_2$  contains all sets in  $S_1$  that do contain the element  $E$ , while  $S_3$  contains all sets in  $S_1$  that do not contain the element  $E$ . Thus,

$$S_2 = \{\{AE\}, \{BE\}, \{CE\}, \{DE\}\}$$

and

$$S_3 = \{\{AB\}, \{AC\}, \{AD\}, \{BC\}, \{BD\}, \{CD\}\}.$$

Each set in  $S_2$  must contain  $E$  and then contains one other element from  $S$ . Since there are 4 other elements in  $S$  that we can choose along with  $E$ ,  $|S_2| = \binom{4}{1}$ . Each set in  $S_3$  contains 2 elements from the set  $\{A, B, C, D\}$ , and thus there are  $\binom{4}{2}$  ways to choose such a subset. But  $S_1 = S_2 \cup S_3$  and  $S_2$  and  $S_3$  are disjoint, and so, by the sum principle, Equation 1.7 must hold.

We now give a proof for general  $n$  and  $k$ .

**Theorem 1.3** *If  $n$  and  $k$  are integers with  $n > 0$  and  $0 < k < n$ , then*

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

**Proof:** The formula says that the number of  $k$ -element subsets of an  $n$ -element set is the sum of two numbers. As in our example, we will apply the sum principle. To apply it, we need to represent the set of  $k$ -element subsets of an  $n$ -element set as a union of two other disjoint sets. Suppose our  $n$ -element set is  $S = \{x_1, x_2, \dots, x_n\}$ . Then we wish to take  $S_1$ , say, to be the  $\binom{n}{k}$ -element set of all  $k$ -element subsets of  $S$  and partition it into two disjoint sets of  $k$ -element subsets,  $S_2$  and  $S_3$ , where the sizes of  $S_2$  and  $S_3$  are  $\binom{n-1}{k-1}$  and  $\binom{n-1}{k}$  respectively. We can do this as follows. Note that  $\binom{n-1}{k}$  stands for the number of  $k$  element subsets of the first  $n-1$  elements  $x_1, x_2, \dots, x_{n-1}$  of  $S$ . Thus we can let  $S_3$  be the set of  $k$ -element subsets of  $S$  that don't contain  $x_n$ . Then the only possibility for  $S_2$  is the set of  $k$ -element subsets of  $S$  that *do* contain  $x_n$ . How can we see that the number of elements of this set  $S_2$  is  $\binom{n-1}{k-1}$ ? By observing that removing  $x_n$  from each of the elements of  $S_2$  gives a  $(k-1)$ -element subset of  $S' = \{x_1, x_2, \dots, x_{n-1}\}$ . Further each  $(k-1)$ -element subset of  $S'$  arises in this way from one and only one  $k$ -element subset of  $S$  containing  $x_n$ . Thus the number of elements of  $S_2$  is the number of  $(k-1)$ -element subsets of  $S'$ , which is  $\binom{n-1}{k-1}$ . Since  $S_2$  and  $S_3$  are two disjoint sets whose union is  $S$ , the sum principle shows that the number of elements of  $S$  is  $\binom{n-1}{k-1} + \binom{n-1}{k}$ . ■

Notice that in our proof, we used a bijection that we did not explicitly describe. Namely, there is a bijection  $f$  between  $S_3$  (the  $k$ -element sets of  $S$  that contain  $x_n$ ) and the  $(k-1)$ -element subsets of  $S'$ . For any subset  $K$  in  $S_3$ , We let  $f(K)$  be the set we obtain by removing  $x_n$  from  $K$ . It is immediate that this is a bijection, and so the bijection principle tells us that the size of  $S_3$  is the size of the set of all subsets of  $S'$ .

## The Binomial Theorem

**Exercise 1.3-3** What is  $(x + y)^3$ ? What is  $(x + 1)^4$ ? What is  $(2 + y)^4$ ? What is  $(x + y)^4$ ?

The number of  $k$ -element subsets of an  $n$ -element set is called a *binomial coefficient* because of the role that these numbers play in the algebraic expansion of a binomial  $x + y$ . The **Binomial Theorem** states that

**Theorem 1.4 (Binomial Theorem)** For any integer  $n \geq 0$

$$(x + y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n, \quad (1.8)$$

or in summation notation,

$$(x + y)^n = \sum_{i=0}^n \binom{n}{i} x^{n-i} y^i .$$

Unfortunately when most people first see this theorem, they do not have the tools to see easily why it is true. Armed with our new methodology of using subsets to prove algebraic identities, we can give a proof of this theorem.

Let us begin by considering the example  $(x + y)^3$  which by the binomial theorem is

$$(x + y)^3 = \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3 \quad (1.9)$$

$$= x^3 + 3x^2y + 3xy^2 + y^3 . \quad (1.10)$$

Suppose that we did not know the binomial theorem but still wanted to compute  $(x + y)^3$ . Then we would write out  $(x + y)(x + y)(x + y)$  and perform the multiplication. Probably we would multiply the first two terms, obtaining  $x^2 + 2xy + y^2$ , and then multiply this expression by  $x + y$ . Notice that by applying distributive laws you get

$$(x + y)(x + y) = (x + y)x + (x + y)y = xx + xy + yx + y. \quad (1.11)$$

We could use the commutative law to put this into the usual form, but let us hold off for a moment so we can see a pattern evolve. To compute  $(x + y)^3$ , we can multiply the expression on the right hand side of Equation 1.11 by  $x + y$  using the distributive laws to get

$$(xx + xy + yx + yy)(x + y) = (xx + xy + yx + yy)x + (xx + xy + yx + yy)y \quad (1.12)$$

$$= xxx + xyx + yxx + yxx + xsy + xyy + yxy + yyy \quad (1.13)$$

Each of these 8 terms that we got from the distributive law may be thought of as a product of terms, one from the first binomial, one from the second binomial, and one from the third binomial. Multiplication is commutative, so many of these products are the same. In fact, we have one  $xxx$  or  $x^3$  product, three products with two  $x$ 's and one  $y$ , or  $x^2y$ , three products with one  $x$  and two  $y$ 's, or  $xy^2$  and one product which becomes  $y^3$ . Now look at Equation 1.9, which summarizes this process. There are  $\binom{3}{0} = 1$  way to choose a product with 3  $x$ 's and 0  $y$ 's,  $\binom{3}{1} = 3$  way to choose a product with 2  $x$ 's and 1  $y$ , etc. Thus we can understand the binomial theorem

as counting the subsets of our binomial factors from which we choose a  $y$ -term to get a product with  $k$   $y$ 's in multiplying a string of  $n$  binomials.

Essentially the same explanation gives us a proof of the binomial theorem. Note that when we multiplied out three factors of  $(x + y)$  using the distributive law but not collecting like terms, we had a sum of eight products. Each factor of  $(x + y)$  doubles the number of summands. Thus when we apply the distributive law as many times as possible (without applying the commutative law and collecting like terms) to a product of  $n$  binomials all equal to  $(x + y)$ , we get  $2^n$  summands. Each summand is a product of a length  $n$  list of  $x$ 's and  $y$ 's. In each list, the  $i$ th entry comes from the  $i$ th binomial factor. A list that becomes  $x^{n-k}y^k$  when we use the commutative law will have a  $y$  in  $k$  of its places and an  $x$  in the remaining places. The number of lists that have a  $y$  in  $k$  places is thus the number of ways to select  $k$  binomial factors to contribute a  $y$  to our list. But the number of ways to select  $k$  binomial factors from  $n$  binomial factors is simply  $\binom{n}{k}$ , and so that is the coefficient of  $x^{n-k}y^k$ . This proves the binomial theorem.

Applying the Binomial Theorem to the remaining questions in Exercise 1.3-3 gives us

$$\begin{aligned}(x + 1)^4 &= x^4 + 4x^3 + 6x^2 + 4x + 1 \\ (2 + y)^4 &= 16 + 32y + 24y^2 + 8y^3 + y^4 \text{ and} \\ (x + y)^4 &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.\end{aligned}$$

### Labeling and trinomial coefficients

**Exercise 1.3-4** Suppose that I have  $k$  labels of one kind and  $n - k$  labels of another. In how many different ways may I apply these labels to  $n$  objects?

**Exercise 1.3-5** Show that if we have  $k_1$  labels of one kind,  $k_2$  labels of a second kind, and  $k_3 = n - k_1 - k_2$  labels of a third kind, then there are  $\frac{n!}{k_1!k_2!k_3!}$  ways to apply these labels to  $n$  objects.

**Exercise 1.3-6** What is the coefficient of  $x^{k_1}y^{k_2}z^{k_3}$  in  $(x + y + z)^n$ ?

Exercise 1.3-4 and Exercise 1.3-5 can be thought of as immediate applications of binomial coefficients. For Exercise 1.3-4, there are  $\binom{n}{k}$  ways to choose the  $k$  objects that get the first label, and the other objects get the second label, so the answer is  $\binom{n}{k}$ . For Exercise 1.3-5, there are  $\binom{n}{k_1}$  ways to choose the  $k_1$  objects that get the first kind of label, and then there are  $\binom{n-k_1}{k_2}$  ways to choose the objects that get the second kind of label. After that, the remaining  $k_3 = n - k_1 - k_2$  objects get the third kind of label. The total number of labellings is thus, by the product principle, the product of the two binomial coefficients, which simplifies as follows.

$$\begin{aligned}\binom{n}{k_1} \binom{n-k_1}{k_2} &= \frac{n!}{k_1!(n-k_1)!} \frac{(n-k_1)!}{k_2!(n-k_1-k_2)!} \\ &= \frac{n!}{k_1!k_2!(n-k_1-k_2)!} \\ &= \frac{n!}{k_1!k_2!k_3!}.\end{aligned}$$

A more elegant approach to Exercise 1.3-4, Exercise 1.3-5, and other related problems appears in the next section.

Exercise 1.3-6 shows how Exercise 1.3-5 applies to computing powers of trinomials. In expanding  $(x + y + z)^n$ , we think of writing down  $n$  copies of the trinomial  $x + y + z$  side by side, and applying the distributive laws until we have a sum of terms each of which is a product of  $x$ 's,  $y$ 's and  $z$ 's. How many such terms do we have with  $k_1$   $x$ 's,  $k_2$   $y$ 's and  $k_3$   $z$ 's? Imagine choosing  $x$  from some number  $k_1$  of the copies of the trinomial, choosing  $y$  from some number  $k_2$ , and  $z$  from the remaining  $k_3$  copies, multiplying all the chosen terms together, and adding up over all ways of picking the  $k_i$ s and making our choices. Choosing  $x$  from a copy of the trinomial “labels” that copy with  $x$ , and the same for  $y$  and  $z$ , so the number of choices that yield  $x^{k_1}y^{k_2}z^{k_3}$  is the number of ways to label  $n$  objects with  $k_1$  labels of one kind,  $k_2$  labels of a second kind, and  $k_3$  labels of a third. Notice that this requires that  $k_3 = n - k_1 - k_2$ . By analogy with our notation for a binomial coefficient, we define the *trinomial coefficient*  $\binom{n}{k_1, k_2, k_3}$  to be  $\frac{n!}{k_1!k_2!k_3!}$  if  $k_1 + k_2 + k_3 = n$  and 0 otherwise. Then  $\binom{n}{k_1, k_2, k_3}$  is the coefficient of  $x^{k_1}y^{k_2}z^{k_3}$  in  $(x + y + z)^n$ . This is sometimes called the *trinomial theorem*.

### Important Concepts, Formulas, and Theorems

1. *Pascal Relationship*. The Pascal Relationship says that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k},$$

whenever  $n > 0$  and  $0 < k < n$ .

2. *Pascal's Triangle*. Pascal's Triangle is the triangular array of numbers we get by putting ones in row  $n$  and column 0 and in row  $n$  and column  $n$  of a table for every positive integer  $n$  and then filling the remainder of the table by letting the number in row  $n$  and column  $j$  be the sum of the numbers in row  $n - 1$  and columns  $j - 1$  and  $j$  whenever  $0 < j < n$ .

3. *Binomial Theorem*. The **Binomial Theorem** states that for any integer  $n \geq 0$

$$(x + y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n,$$

or in summation notation,

$$(x + y)^n = \sum_{i=0}^n \binom{n}{i} x^{n-i} y^i.$$

4. *Labeling*. The number of ways to apply  $k$  labels of one kind and  $n - k$  labels of another kind to  $n$  objects is  $\binom{n}{k}$ .
5. *Trinomial coefficient*. We define the *trinomial coefficient*  $\binom{n}{k_1, k_2, k_3}$  to be  $\frac{n!}{k_1!k_2!k_3!}$  if  $k_1 + k_2 + k_3 = n$  and 0 otherwise.
6. *Trinomial Theorem*. The coefficient of  $x^i y^j z^k$  in  $(x + y + z)^n$  is  $\binom{n}{i, j, k}$ .

## Problems

1. Find  $\binom{12}{3}$  and  $\binom{12}{9}$ . What can you say in general about  $\binom{n}{k}$  and  $\binom{n}{n-k}$ ?
2. Find the row of the Pascal triangle that corresponds to  $n = 8$ .
3. Find the following
  - a.  $(x + 1)^5$
  - b.  $(x + y)^5$
  - c.  $(x + 2)^5$
  - d.  $(x - 1)^5$
4. Carefully explain the proof of the binomial theorem for  $(x + y)^4$ . That is, explain what each of the binomial coefficients in the theorem stands for and what powers of  $x$  and  $y$  are associated with them in this case.
5. If I have ten distinct chairs to paint in how many ways may I paint three of them green, three of them blue, and four of them red? What does this have to do with labellings?
6. When  $n_1, n_2, \dots, n_k$  are nonnegative integers that add to  $n$ , the number  $\frac{n!}{n_1!n_2!\dots n_k!}$  is called a *multinomial coefficient* and is denoted by  $\binom{n}{n_1, n_2, \dots, n_k}$ . A polynomial of the form  $x_1 + x_2 + \dots + x_k$  is called a multinomial. Explain the relationship between powers of a multinomial and multinomial coefficients. This relationship is called the Multinomial Theorem.
7. Give a bijection that proves your statement about  $\binom{n}{k}$  and  $\binom{n}{n-k}$  in Problem 1 of this section.
8. In a Cartesian coordinate system, how many paths are there from the origin to the point with integer coordinates  $(m, n)$  if the paths are built up of exactly  $m + n$  horizontal and vertical line segments each of length one?
9. What is the formula we get for the binomial theorem if, instead of analyzing the number of ways to choose  $k$  distinct  $y$ 's, we analyze the number of ways to choose  $k$  distinct  $x$ 's?
10. Explain the difference between choosing four disjoint three element sets from a twelve element set and labelling a twelve element set with three labels of type 1, three labels of type two, three labels of type 3, and three labels of type 4. What is the number of ways of choosing three disjoint four element subsets from a twelve element set? What is the number of ways of choosing four disjoint three element subsets from a twelve element set?
11. A 20 member club must have a President, Vice President, Secretary and Treasurer as well as a three person nominations committee. If the officers must be different people, and if no officer may be on the nominating committee, in how many ways could the officers and nominating committee be chosen? Answer the same question if officers may be on the nominating committee.
12. Prove Equation 1.6 by plugging in the formula for  $\binom{n}{k}$ .

13. Give two proofs that

$$\binom{n}{k} = \binom{n}{n-k}.$$

14. Give at least two proofs that

$$\binom{n}{k} \binom{k}{j} = \binom{n}{j} \binom{n-j}{k-j}.$$

15. Give at least two proofs that

$$\binom{n}{k} \binom{n-k}{j} = \binom{n}{j} \binom{n-j}{k}.$$

16. You need not compute all of rows 7, 8, and 9 of Pascal's triangle to use it to compute  $\binom{9}{6}$ . Figure out which entries of Pascal's triangle not given in Table 2 you actually need, and compute them to get  $\binom{9}{6}$ .

17. Explain why

$$\sum_{i=0}^n (-1)^i \binom{n}{i} = 0$$

18. Apply calculus and the binomial theorem to  $(1+x)^n$  to show that

$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \dots = n2^{n-1}.$$

19. True or False:  $\binom{n}{k} = \binom{n-2}{k-2} + \binom{n-2}{k-1} + \binom{n-2}{k}$ . If True, give a proof. If false, give a value of  $n$  and  $k$  that show the statement is false, find an analogous true statement, and prove it.