1.4 Equivalence Relations and Counting

Counting using Equivalence Classes

Consider again the example from Section 1.2 in which we wanted to count the number of 3 element subsets of a four element set. To do so, we first formed all possible lists of $k = 3$ distinct elements chosen from an $n = 4$ element set. (See Equation 1.4.) The number of lists of k distinct elements is $n^{\underline{k}} = n!/(n-k)!$. We then observed that two lists are equivalent as sets, if one can be obtained by rearranging (or "permuting") the other. This process divides the lists up into classes, called *equivalence classes*, each of size $k!$. Returning to our example in Section 1.2, we noted that one such equivalence class was

$$\{134, 143, 314, 341, 413, 431\} .$$

The other three are

$$\{234, 243, 324, 342, 423, 432\} ,$$

$$\{132, 123, 312, 321, 213, 231\} ,$$

and

$$\{124, 142, 214, 241, 412, 421\} .$$

The product principle told us that if q is the number of such equivalence class, if each equivalence class has $k!$ elements, and the entire set of lists has $n!/(n-k)!$ element, then we must have that

$$qk! = n!/(n-k)! .$$

Dividing, we solve for q and get an expression for the number of k element subsets of an n element set. In fact, this is how we proved Theorem 1.2.

A principle that helps in learning and understanding mathematics is that if we have a mathematical result that shows a certain symmetry, it often helps our understanding to find a proof that reflects this symmetry. We call this the *Symmetry Principle*. The proof above does not account for the symmetry of the $k!$ term and the $(n-k)!$ term in the expression $\frac{n!}{k!(n-k)!}$. This symmetry arises because choosing a k element subset is equivalent to choosing the $(n-k)$-element subset of elements we don't want. In Exercise 1.4-4, we saw that the binomial coefficient $\binom{n}{k}$ also counts the number of ways to label n objects, say with the labels "in" and "out," so that we have k "ins" and therefore $n-k$ "outs." For each labelling, the k objects that get the label "in" are in our subset. Here is a new proof that the number of labellings is $n!/k!(n-k)!$ that explains the symmetry.

Suppose we have m ways to assign k blue and $n-k$ red labels to n elements. From each labeling, we can create a number of lists, using the convention of listing the k blue elements first and the remaining $n-k$ red elements last. For example, suppose we are considering the number of ways to label 3 elements blue (and 2 red) from a five element set $\{A, B, C, D, E\}$. Consider the particular labelling in which A, B, and D are labelled blue and C and E are labelled red. Which lists correspond to this labelling? They are

$$\begin{array}{cccccc} ABDCE & ABDEC & ADBCE & ADBEC & BADCE & BADEC \\ BDACE & BDAEC & DABCE & DABEC & DBACE & DBAEC \end{array}$$

that is, all lists in which A, B, and D precede C and E. Since there are 3! ways to arrange A, B, and D, and 2! ways to arrange C and E, by the product principal, there are $3!2! = 12$ lists in which A, B, and D precede C and E. For each of the q ways to construct a labelling, we could find a similar set of 12 lists that are associated with that labelling. Since *every* possible list of 5 elements will appear exactly once via this process, and since there are $5! = 120$ five-element lists overall, we must have by the product principle that

$$q \cdot 12 = 120, \tag{1.14}$$

or that $q = 10$. This agrees with our previous calculations of $\binom{5}{3} = 10$ for the number of ways to label 5 items so that 3 are blue and 2 are red.

Generalizing, we let q be the number of ways to label n objects with k blue labels and $n-k$ red labels. To create the lists associated with a labelling, we list the blue elements first and then the red elements. We can mix the k blue elements among themselves, and we can mix the $n-k$ red elements among themselves, giving us $k!(n-k)!$ lists consisting of first the elements with a blue label followed by the elements with a red label. Since we can choose to label any k elements blue, each of our lists of n distinct elements arises from some labelling in this way. Each such list arises from only one labelling, because two different labellings will have a different first k elements in any list that corresponds to the labelling. Each such list arises only once from a given labelling, because two different lists that correspond to the same labelling differ by a permutation of the first k places or the last $n-k$ places or both. Therefore, by the product principle, $qk!(n-k)!$ is the number of lists we can form with n distinct objects, and this must equal $n!$. This gives us

$$qk!(n-k)! = n!,$$

and division gives us our original formula for q. Recall that our proof of the formula we had in Exercise 1.4-5 did not explain why the product of three factorials appeared in the denominator, it simply proved the formula was correct. With this idea in hand, we could now explain *why* the product in the denominator of the formula in Exercise 1.4-5 for the number of labellings with three labels is what it is, and could generalize this formula to four or more labels.

Equivalence Relations

The process above divided the set of all $n!$ lists of n distinct elements into classes (another word for sets) of lists. In each class, all the lists are mutually equivalent, with respect to labeling with two labels. More precisely, two lists of the n objects are equivalent for defining labellings if we get one from the other by mixing the first k elements among themselves and mixing the last $n-k$ elements among themselves. Relating objects we want to count to sets of lists (so that each object corresponds to an set of equivalent lists) is a technique we can use to solve a wide variety of counting problems. (This is another example of abstraction.)

A relationship that divides a set up into mutually exclusive classes is called an **equivalence relation**.[7] Thus, if

$$S = S_1 \cup S_2 \cup \ldots \cup S_m$$

[7]The usual mathematical approach to equivalence relations, which we shall discuss in the exercises, is different from the one given here. Typically, one sees an equivalence relation defined as a reflexive (everything is related to itself), symmetric (if x is related to y, then y is related to x), and transitive (if x is related to y and y is related to z, then x is related to z) relationship on a set X. Examples of such relationships are equality (on any set), similarity (on a set of triangles), and having the same birthday as (on a set of people). The two approaches are equivalent, and we haven't found a need for the details of the other approach in what we are doing in this course.

and $S_i \cap S_j = \emptyset$ for all i and j with $i \neq j$, then the relationship that says any two elements $x \in S$ and $y \in S$ are equivalent if and only if they lie in the same set S_i is an equivalence relation. The sets S_i are called *equivalence classes*, and, as we noted in Section 1.1 the family $S_1, S_2, \ldots, S_m$ is called a **partition** of S. One partition of the set $S = \{a, b, c, d, e, f, g\}$ is $\{a, c\}$, $\{d, g\}$, $\{b, e, f\}$. This partition corresponds to the following (boring) equivalence relation: a and c are equivalent, d and g are equivalent, and b, e, and f are equivalent. A slightly less boring equivalence relation is that two letters are equivalent if typographically, their top and bottom are at the same height. This give the partition $\{a, c, e\}$, $\{b, d\}$, $\{f\}$, $\{g\}$.

Exercise 1.4-1 On the set of integers between 0 and 12 inclusive, define two integers to be related if they have the same remainder on division by 3. Which numbers are related to 0? to 1? to 2? to 3? to 4?. Is this relationship an equivalence relation?

In Exercise 1.4-1, the numbers related to 0 are the set $\{0, 3, 6, 9, 12\}$, those related to 1 are $\{1, 4, 7, 10\}$, those related to 2 are $\{2, 5, 8, 11\}$, those related to 3 are $\{0, 3, 6, 9, 12\}$, those related to 4 are $\{1, 4, 7, 10\}$. A little more precisely, a number is related to one of 0, 3, 6, 9, or 12, if and only if it is in the set $\{0, 3, 6, 9, 12\}$, a number is related to 1, 4, 7, or 10 if and only if it is in the set $\{1, 4, 7, 10\}$ and a number is related to 2, 5, 8, or 11 if and only if it is in the set $\{2, 5, 8, 11\}$. Therefore the relationship is an equivalence relation.

The quotient principle

In Exercise 1.4-1 the equivalence classes had two different sizes. In the examples of counting labellings and subsets that we have seen so far, all the equivalence classes had the same size, and this was very important. The principle we have been using to count subsets and labellings is the following theorem. We will call this principle the **Quotient Principle**.

Theorem 1.5 (Quotient principle) *If an equivalence relation on a p-element set S has q classes each of size r, then $q = p/r$.*

Proof: By the product principle, $s = mt$, and so $m = s/t$. ∎

Another statement of the quotient principle that uses the idea of a partition is

Principle 1.6 (Quotient principle.) *If we can partition a set of size p into q blocks of size r, then $q = p/r$.*

Returning to our example of 3 blue and 2 red labels, $s = 5! = 120$, $t = 12$ and so by Theorem 1.5,

$$m = \frac{s}{t} = \frac{120}{12} = 10 \ .$$

Equivalence class counting

We now give several examples of the use of Theorem 1.5.

Exercise 1.4-2 When four people sit down at a round table to play cards, two lists of their four names are equivalent as seating charts if each person has the same person to the right in both lists[8]. (The person to the right of the person in position 4 of the list is the person in position 1). We will use Theorem 1.5 to count the number of possible ways to seat the players. We will take our set S to be the set of all 4-element permutations of the four people, i.e., the set of all lists of the four people.

(a) How many lists are equivalent to a given one?

(b) What are the lists equivalent to ABCD?

(c) Is the relationship of equivalence an equivalence relation?

(d) Use Theorem 1.5 to compute the number of equivalence classes, and hence, the number of possible ways to seat the players.

Exercise 1.4-3 We wish to count the number of ways to attach n distinct beads to the corners of a regular n-gon (or string them on a necklace). We say that two lists of the n beads are equivalent if each bead is adjacent to exactly the same beads in both lists. (The first bead in the list is considered to be adjacent to the last.)

- How does this exercise differ from the previous exercise?
- How many lists are in an equivalence class?
- How many equivalence classes are there?

In Exercise 1.4-2, suppose we have named the places at the table north, east, south, and west. Given a list we get an equivalent one in two steps. First we observe that we have four choices of people to sit in the north position. Then there is one person who can sit to this person's right, one who can be next on the right, and one who can be the following on on the right, all determined by the original list. Thus there are exactly four lists equivalent to a given one, including that given one. The lists equivalent to ABCD are ABCD, BCDA, CDAB, and DABC. This shows that two lists are equivalent if and only if we can get one from the other by moving everyone the same number of places to the right around the table (or we can get one from the other moving everyone the same number of places to the left around the table). From this we can see we have an equivalence relation, because each list is in one of these sets of four equivalent lists, and if two lists are equivalent, they are right or left shifts of each other, and we've just observed that all right and left shifts of a given list are in the same class. This means our relationship divides the set of all lists of the four names into equivalence classes each of size four. There are a total of $4! = 24$ lists of four distinct names, and so by Theorem 1.5 we have $4!/4 = 3! = 6$ seating arrangements.

Exercise 1.4-3 is similar in many ways to Exercise 1.4-2, but there is one significant difference. We can visualize the problem as one of dividing lists of n distinct beads up into equivalence classes, but now two lists are equivalent if each bead is adjacent to exactly the same beads in both of them. Suppose we number the vertices of our polygon as 1 through n clockwise. Given a list, we can count the equivalent lists as follows. We have n choices for which bead to put in position 1. Then either of the two beads adjacent to it in the given list can go in position 2. But now, only one bead can go in position 3, because the other bead adjacent to position 2 is already in position

[8]Think of the four places at the table as being called north, east, south, and west, or numbered 1-4. Then we get a list by starting with the person in the north position (position 1), then the person in the east position (position 2) and so on clockwise

1. We can continue in this way to fill in the rest of the list. For example, with $n = 4$, the lists ABCD, ADBC, DABC DCBA, CDBA, CBAD, BCDA,and BADC are all equivalent. Notice the first, third , fifth and seventh lists are obtained by shifting the beads around the polygon, as are the second, fourth, sixth and eighth. Also note that the fourth list is the reversal of the first, the fifth is the reversal of the second, and so on. Rotating a necklace in space corresponds to shifting the letters in the list. Flipping a necklace over in space corresponds to reversing the order of a list. There will always be $2n$ lists we can get by shifting and reversing shifts of a list. The lists equivalent to a given one consist of everything we can get from the given list by rotations and reversals. Thus the relationship of every bead being adjacent to the same beads divides the set of lists of beads into disjoint sets. These sets, which have size $2n$, are the equivalence classes of our equivalence relation. Since there are $n!$ lists, Theorem 1.5 says there are

$$\frac{n!}{2n} = \frac{(n-1)!}{2}$$

bead arrangements.

Multisets

Sometimes when we think about choosing elements from a set, we want to be able to choose an element more than once. For example the set of letters of the word "roof" is $\{f, o, r\}$. However it is often more useful to think of the of the *multiset* of letters, which in this case is $\{\{f, o, o, r\}\}$. We use the double brackets to distinguish a multiset from a set. We can specify a *multiset* chosen from a set S by saying how many times each of its elements occurs. If S is the set of English letters, the "multiplicity" function for roof is given by $m(f) = 1$, $m(o) = 2$, $m(r) = 1$, and $m(\text{letter}) = 0$ for every other letter. In a multiset, order is not important, that is the multiset $\{\{r, o, f, o\}\}$ is equivalent to the multiset $\{\{f, o, o, r\}\}$. We know that this is the case, because they each have the same multiplicity function. We would like to say that the size of $\{\{f, o, o, r\}\}$ is 4, so we define the*size* of a multiset to be the sum of the multiplicities of its elements.

Exercise 1.4-4 Explain how placing k identical books onto the n shelves of a bookcase can be thought of as giving us a k-element multiset of the shelves of the bookcase. Explain how distributing k identical apples to n children can be thought of as giving us a k-element multiset of the children.

In Exercise 1.4-4 we can think of the multiplicity of a bookshelf as the number of books it gets and the multiplicity of a child as the number of apples the child gets. In fact, this idea of distribution of identical objects to distinct recipients gives a great mental model for a multiset chosen from a set S. Namely, to determine a k-element multiset chosen from S form S, we "distribute" k identical objects to the elements of S and the number of objects an element x gets is the multiplicity of x.

Notice that it makes no sense to ask for the number of multisets we may choose from a set with n elements, because $\{\{A\}\}$, $\{\{A, A\}\}$, $\{\{A, A, A\}\}$, and so on are infinitely many multisets chosen from the set $\{A\}$. However it does make sense to ask for the number of k-element multisets we can choose from an n-element set. What strategy could we employ to figure out this number? To count k-element subsets, we first counted k-element permutations, and then divided by the number of different permutations of the same set. Here we need an analog of permutations that allows repeats. A natural idea is to consider lists with repeats. After all, one way to describe a multiset is to list it, and there could be many different orders for listing a multiset. However the two element multiset $\{\{A, A\}\}$ can be listed in just one way, while the two element multiset $\{\{A, B\}\}$ can be listed in two ways. When we counted k-element subsets of an n-element set by using the quotient principle, it was essential that each k-element set corresponded to the same number (namely $k!$) of permutations (lists), because we were using the reasoning behind the quotient principle to do our counting here. So if we hope to use similar reasoning, we can't apply it to lists because different k-element multisets can correspond to different numbers of lists.

Suppose, however, we could count the number of ways to arrange k distinct books on the n shelves of a bookcase. We can still think of the multiplicity of a shelf as being the number of books on it. However, many different arrangements of distinct books will give us the same multiplicity function. In fact, any way of mixing the books up among themselves that does not change the number of books on each shelf will give us the same multiplicities. But the number of ways to mix the books up among themselves is the number of permutations of the books, namely $k!$. Thus it looks like we have an equivalence relation on the arrangements of distinct books on a bookshelf such that

1. Each equivalence class has $k!$ elements, and
2. There is a bijection between the equivalence classes and k-element multisets of the n shelves.

Thus if we can compute the number of ways to arrange k *distinct* books on the n shelves of a bookcase, we should be able to apply the quotient principle to compute the number of k-element multisets of an n-element set.

The bookcase arrangement problem.

Exercise 1.4-5 We have k books to arrange on the n shelves of a bookcase. The order in which the books appear on a shelf matters, and each shelf can hold all the books. We will assume that as the books are placed on the shelves they are moved as far to the left as they will go so that all that matters is the order in which the books appear and not the actual places where the books sit. When book i is placed on a shelf, it can go between two books already there or to the left or right of all the books on that shelf.

(a) Since the books are distinct, we may think of a first, second, third, etc. book. In how many ways may we place the first book on the shelves.

(b) Once the first book has been placed, in how many ways may the second book be placed?

(c) Once the first two books have been placed, in how many ways may the third book be placed?

(d) Once $i - 1$ books have been placed, book i can be placed on any of the shelves to the left of any of the books already there, but there are some additional ways in which it may be placed. In how many ways in total may book i be placed?

(e) In how many ways may k distinct books be place on n shelves in accordance with the constraints above?

In Exercise 1.4-5 there are n places where the first book can go, namely on the left side of any shelf. Then the next book can go in any of the n places on the far left side of any shelf, or it can go to the right of book one. Thus there are $n + 1$ places where book 2 can go. At first, placing book three appears to be more complicated, because we could create two different patterns by placing the first two books. However book 3 could go to the far left of any shelf or to the immediate right of any of the books already there. (Notice that if book 2 and book 1 are on shelf 3 in that order, putting book 3 to the immediate right of book 2 means putting it between book 2 and book 1.) Thus in any case, there are n+2 ways to place book 3. Similarly, once $i - 1$ books have been placed, there are $n + i - 1$ places where we can place book i. It can go at the far left of any of the n shelves or to the immediate right of any of the $i - 1$ books that we have already placed. Thus the number of ways to place k distinct books is

$$n(n+1)(n+2)\cdots(n+k-1) = \prod_{i=1}^{k}(n+i-1) = \prod_{j=0}^{k-1}(n+j) = \frac{(n+k-1)!}{(n-1)!}. \tag{1.15}$$

The Pi notation (also known as the product notation) we introduced for the product in Equation 1.15 is completely analogous to the Sigma notation we have learned to use for summation. The specific product that arose in Equation 1.15 is called a *rising factorial power*. It has a notation (also introduced by Don Knuth) analogous to that for the falling factorial notation. Namely, we write

$$n^{\overline{k}} = n(n+1)\cdots(n+k-1) = \prod_{i=1}^{k}(n+i-1).$$

This is the product of k successive numbers beginning with n.

Since the last expression in Equation 1.15 is quotient of two factorials it is natural to ask whether it is counting equivalence classes of an equivalence relation. If so, the set on which the relation is defined has size $(n + k - 1)!$. Thus it might be all lists or permutations of $n + k - 1$ distinct objects. The size of an equivalence class is $(n-1)!$ and so what makes two lists equivalent might be permuting $n - 1$ of the objects among themselves. Can we find such an interpretation?

Exercise 1.4-6 In how many ways may we arrange k distinct books and $n - 1$ identical blocks of wood in a straight line?

Exercise 1.4-7 How does Exercise 1.4-6 relate to arranging books on the shelves of a bookcase?

In Exercise 1.4-6, if we tape numbers to the wood so that so that the pieces of wood are distinguishable, there are $n + k - 1$ arrangements of the books and wood. But since the pieces of wood are actually indistinguishable, $(n - 1)!$ of these arrangements are equivalent. Thus by the quotient principle there are $(n + k - 1)!/(n - 1)!$ arrangements. Such an arrangement allows us to put the books on the shelves as follows: put all the books before the first piece of wood on shelf 1, all the books between the first and second on shelf 2, and so on until you put all the books after the last piece of wood on shelf n.

The number of k-element multisets of an n-element set

We now define two bookcase arrangements of k books on n shelves to be equivalent if we get one from the other by permuting the books among themselves. Thus if two arrangements put the same number of books on each shelf they are put into the same class by this relationship. On the other hand, if two arrangements put a different number of books on at least one shelf, they are not equivalent, and therefore they are put into different classes by this relationship. Thus the classes into which this relationship divides the the arrangements are disjoint and partition the set of all arrangements. Each class has $k!$ arrangements in it. The set of all arrangements has $n^{\overline{k}}$ arrangements in it. This leads to the following theorem.

Theorem 1.6 *The number of k-element multisets chosen from an n-element set is*

$$\frac{n^{\overline{k}}}{k!} = \binom{n+k-1}{k}.$$

Proof: The relationship on bookcase arrangements that two relationships are equivalent if and only if we get one from the other by permuting the books is an equivalence relation. The set of all arrangements has $n^{\overline{k}}$ elements, and the number of elements in an equivalence class is $k!$. By the quotient principle, the number of equivalence classes is $\frac{n^{\overline{k}}}{k!}$. There is a bijection between equivalence classes of bookcase arrangements with k books and multisets with k elements. The second equality follows from the definition of binomial coefficients. ■

The right-hand side of the formula is a binomial coefficient, so it is natural to ask whether there is a way to interpret choosing a k-element *multi*set form an n-element set as choosing a k-element *sub*set of some different $n+k-1$-element set. This illustrates and important principle. When we have a quantity that turns our to be equal to a binomial coefficient, it helps our understanding to interpret it as counting the number of ways to choose a subset of an appropriate size from a set of an appropriate size. We explore this idea for multisets in Problem 8 in this section.

Important Concepts, Formulas, and Theorems

1. *Symmetry Principle.* If we have a mathematical result that shows a certain symmetry, it often helps our understanding to find a proof that reflects this symmetry.

2. *Partition.* Given a set S of items, a *partition* of S consists of m sets $S_1, S_2, \ldots, S_m$, sometimes called *blocks* so that $S_1 \cup S_2 \cup \cdots \cup S_m = S$ and for each i and j with $i \neq j$, $S_i \cap S_j = \emptyset$.

3. *Equivalence relation. Equivalence class.* A relationship that partitions a set up into mutually exclusive classes is called an **equivalence relation**. Thus if $S = S_1 \cup S_2 \cup \ldots \cup S_m$ is a partition of S, the relationship that says any two elements $x \in S$ and $y \in S$ are equivalent if and only if they lie in the same set S_i is an equivalence relation. The sets S_i are called *equivalence classes*

4. *Quotient principle.* The **quotient principle** says that if we can partition a set of p objects up into q classes of size r, then $q = p/r$. Equivalently, if an equivalence relation on a set of size p has q equivalence classes of size r, then $q = p/r$. The quotient principle is frequently used for counting the number of equivalence classes of an equivalence relation. When we have a quantity that is a quotient of two others, it is often helpful to our understanding to find a way to use the quotient principle to explain why we have this quotient.

5. *Multiset.* A multiset is similar to a set except that each item can appear multiple times. We can specify a *multiset* chosen from a set S by saying how many times each of its elements occurs.

6. *Choosing k-element multisets.* The number of k-element multisets that can be chosen from an n-element set is
$$\frac{(n+k-1)!}{k!(n-1)!} = \binom{n+k-1}{k}.$$
This is sometimes called the formula for "combinations!with repetitions."

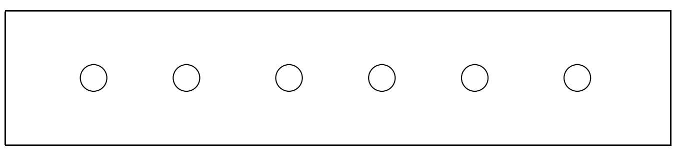
7. *Product notation, Pi notation* The notation $\prod_{i=1}^{n} a_i$ means the product of the quantities a_1 through a_i; that is, $a_1 a_2 \cdots a_n$.

8. When we have a quantity that turns out to be a binomial coefficient (or some other formula we recognize) it is often helpful to our understanding to try to interpret the quantity as the result of choosing a subset of a set (or doing whatever the formula that we recognize counts.)

Problems

1. In how many ways may n people be seated around a round table? (Remember, two seating arrangements around a round table are equivalent if everyone is in the same position relative to everyone else in both arrangements.)

2. In how many ways may we embroider n circles of different colors in a row (lengthwise, equally spaced, and centered halfway between the top and bottom edges) on a scarf (as follows)?



3. Use binomial coefficients to determine in how many ways three identical red apples and two identical golden apples may be lined up in a line. Use equivalence class counting (in particular, the quotient principle) to determine the same number.

4. Use multisets to determine the number of ways to pass out k identical apples to n children.

5. In how many ways may n men and n women be seated around a table alternating gender? (Use equivalence class counting!!)

6. In how many ways may we pass out k identical apples to n children if each child must get at least one apple?

7. In how many ways may we place k distinct books on n shelves of a bookcase (all books pushed to the left as far as possible) if there must be at least one book on each shelf?

8. The formula for the number of multisets is $(n+k-1)!$ divided by a product of two other factorials. We seek an explanation using the quotient principle of why this counts multisets. The formula for the number of multisets is also a binomial coefficient, so it should have an interpretation involving choosing k items from $n+k-1$ items. The parts of the problem that follow lead us to these explanations.

(a) In how many ways may we place k red checkers and $n-1$ black checkers in a row?

(b) How can we relate the number of ways of placing k red checkers and $n-1$ black checkers in a row to the number of k-element multisets of an n-element set, say the set $\{1, 2, \ldots, n\}$ to be specific?

(c) How can we relate the choice of k items out of $n+k-1$ items to the placement of red and black checkers as in the previous parts of this problem?

9. How many solutions to the equation $x_1 + x_2 + \cdots x_n = k$ are there with each $x_i \geq 0$?

10. How many solutions to the equation $x_1 + x_2 + \cdots x_n = k$ are there with each $x_i > 0$?

11. In how many ways may n red checkers and $n+1$ black checkers be arranged in a circle? (This number is a famous number called a *Catalan number*.)

12. A standard notation for the number of partitions of an n element set into k classes is $S(n, k)$. $S(0, 0)$ is 1, because technically the empty family of subsets of the empty set is a partition of the empty set, and $S(n, 0)$ is 0 for $n > 0$, because there are no partitions of a nonempty set into no parts. $S(1, 1)$ is 1.

(a) Explain why $S(n, n)$ is 1 for all $n > 0$. Explain why $S(n, 1)$ is 1 for all $n > 0$.

(b) Explain why, for $1 < k < n$, $S(n, k) = S(n-1, k-1) + kS(n-1, k)$.

(c) Make a table like our first table of binomial coefficients that shows the values of $S(n, k)$ for values of n and k ranging from 1 to 6.

13. You are given a square, which can be rotated 90 degrees at a time (i.e. the square has four orientations). You are also given two red checkers and two black checkers, and you will place each checker on one corner of the square. How many lists of four letters, two of which are R and two of which are B, are there? Once you choose a starting place on the square, each list represents placing checkers on the square in clockwise order. Consider two lists to be equivalent if they represent the same arrangement of checkers at the corners of the square, that is, if one arrangement can be rotated to create the other one. Write down the equivalence classes of this equivalence relation. Why can't we apply Theorem 1.5 to compute the number of equivalence classes?

14. The terms "reflexive", "symmetric" and "transitive" were defined in Footnote 2. Which of these properties is satisfied by the relationship of "greater than?" Which of these properties is satisfied by the relationship of "is a brother of?" Which of these properties is satisfied by "is a sibling of?" (You are not considered to be your own brother or your own sibling). How about the relationship "is either a sibling of or is?"

a Explain why an equivalence relation (as we have defined it) is a reflexive, symmetric, and transitive relationship.

b Suppose we have a reflexive, symmetric, and transitive relationship defined on a set S. For each x is S, let $S_x = \{y | y \text{ is related to } x\}$. Show that two such sets S_x and S_y are either disjoint or identical. Explain why this means that our relationship is an equivalence relation (as defined in this section of the notes, not as defined in the footnote).

c Parts b and c of this problem prove that a relationship is an equivalence relation if and only if it is symmetric, reflexive, and transitive. Explain why. (A short answer is most appropriate here.)

15. Consider the following C++ function to compute $\binom{n}{k}$.

```
int pascal(int n, int k)
{
  if (n < k)
    {

      cout << "error: n<k" << endl;
      exit(1);
    }

  if ( (k==0) || (n==k))
    return 1;


  return  pascal(n-1,k-1) + pascal(n-1,k);
}
```

Enter this code and compile and run it (you will need to create a simple main program that calls it). Run it on larger and larger values of n and k, and observe the running time of the program. It should be surprisingly slow. (Try computing, for example, $\binom{30}{15}$.) Why is it so slow? Can you write a different function to compute $\binom{n}{k}$ that is *significantly faster*? Why is your new version faster? (Note: an exact analysis of this might be difficult at this point in the course, it will be easier later. However, you should be able to figure out roughly why this version is so much slower.)

16. Answer each of the following questions with either n^k, $n^{\underline{k}}$, $\binom{n}{k}$, or $\binom{n+k-1}{k}$.

    (a) In how many ways can k different candy bars be distributed to n people (with any person allowed to receive more than one bar)?

    (b) In how many ways can k different candy bars be distributed to n people (with nobody receiving more than one bar)?

    (c) In how many ways can k identical candy bars distributed to n people (with any person allowed to receive more than one bar)?

    (d) In how many ways can k identical candy bars distributed to n people (with nobody receiving more than one bar)?

    (e) How many one-to-one functions f are there from $\{1, 2, \ldots, k\}$ to $\{1, 2, \ldots, n\}$?

    (f) How many functions f are there from $\{1, 2, \ldots, k\}$ to $\{1, 2, \ldots, n\}$?

    (g) In how many ways can one choose a k-element subset from an n-element set?

    (h) How many k-element multisets can be formed from an n-element set?

    (i) In how many ways can the top k ranking officials in the US government be chosen from a group of n people?

    (j) In how many ways can k pieces of candy (not necessarily of different types) be chosen from among n different types?

    (k) In how many ways can k children each choose one piece of candy (all of different types) from among n different types of candy?