Chapter 4

Induction, Graphs and Trees

4.1 Mathematical Induction

Smallest Counter-Examples

We've seen one way of proving statements about infinite universes, namely consider a "generic" member of the universe and try to derive the desired statement about that generic member. When our universe is the universe of integers, or is in a one-to-one correspondence with the integers, there is a second technique we can use.

Recall our our proof of Euclid's Division Theorem (Theorem 2.12), which says that for each pair (k, n) of positive integers, there are nonnegative integers q and r such that $k = nq + r$ and $r < n$. "Among all pairs (k, n) that make it false, choose the smallest k that makes it false. We cannot have $k < n$ because then the statement would be true with $q = 0$, and we cannot have $k = n$ because then the statement is true with $q = 1$ and $r = 0$. This means $k - n$ is a positive number smaller than k, and so there must exist a q and r such that

$$k - n = qn + r, \text{ with } 0 \leq r < n.$$

Thus $k = (q + 1)n + r$, contradicting the assumption that the statement is false, so the only possibility is that the statement is true."

Focus on the sentences "This means $k - n$ is a positive number smaller than k, and so there must exist a q and r such that

$$k - n = qn + r, \text{ with } 0 \leq r < n.$$

Thus $k = (q + 1)n + r, \ldots$." To analyze these sentences, let $p(k, n)$ be the statement "there are nonnegative integers q and r with $0 \leq r < n$ such that $k = nq + r$" The sentences we quoted are a proof that $p(k - n, n) \Rightarrow p(k, n)$. It is this implication that makes the proof work. In outline our proof went like this: we assumed a counter-example with a smallest k existed, then we took advantage of the fact that $p(k', n)$ had to be true for every k' smaller than k, we chose $k' = k - n$, and used the implication $p(k - n, n) \Rightarrow p(k, n)$ to conclude the truth of $p(k, n)$, which we had assumed to be false. This is what gave us our contradiction in our proof by contradiction.

Exercise 4.1-1 In Chapter 1 we learned Gauss's trick for showing that for all positive integers n,

$$1 + 2 + 3 + 4 + ... + n = \frac{n(n+1)}{2}.$$

Use the technique of asserting that if there is a counter-example, there is a smallest counter-example and deriving a contradiction to prove that the sum is $n(n+1)/2$. What implication did you have to prove in the process?

Exercise 4.1-2 For what values of $n \geq 0$ do you think $2^{n+1} \geq n^2 + 2$? Use the technique of asserting there is a smallest counter-example and deriving a contradiction to prove you are right. What implication did you have to prove in the process?

Exercise 4.1-3 Would it make sense to say that if there is a counter example there is a *largest* counter-example and try to base a proof on this? Why or why not?

In Exercise 4.1-1, suppose the formula for the sum is false. Then there must be a smallest n such that the formula does not hold for the sum of the first n positive integers. Thus for any positive integer i smaller than n,

$$1 + 2 + \cdots + i = \frac{i(i+1)}{2}. \tag{4.1}$$

Because $1 = 1 \cdot 2/2$, we know $n > 1$, so that $n - 1$ is one of the positive integers i for which the formula holds. Substituting $n - 1$ for i in Equation 4.1 gives us

$$1 + 2 + \cdots + n - 1 = \frac{(n-1)n}{2}.$$

Adding n to both sides gives

$$\begin{aligned} 1 + 2 + \cdots + n - 1 + n &= \frac{(n-1)n}{2} + n \\ &= \frac{n^2 - n + 2n}{2} \\ &= \frac{n(n+1)}{2}. \end{aligned}$$

Thus n is not a counter-example after all, and therefore there is no counter-example to the formula. Thus the formula holds for all positive integers n. Note that the crucial step was proving that $p(n-1) \Rightarrow p(n)$, where $p(n)$ is the formula

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

In Exercise 4.1-2, let $p(n)$ be the statement that $2^{n+1} \geq n^2 + 2$. Some experimenting with small values of n leads us to believe this statement is true for all nonnegative integers. Thus we want to prove $p(n)$ is true for all nonnegative integers n. For this purpose, we assume that the statement that "$p(n)$ is true for all nonnegative integers n" is false. When a forall statement is false, there must be some n for which it is false. Therefore, there is some smallest nonnegative integer n so that $2^{n+1} \not\geq n^2 + 2$. This means that for all nonnegative integers i with $i < n$, $2^{i+1} \geq i^2 + 2$.

Since we know from our experimentation that $n \neq 0$, we know $n-1$ is a nonnegative integer less than n, so using $n-1$ in place of i, we get

$$2^{(n-1)+1} \geq (n-1)^2 + 2,$$

or

$$\begin{aligned} 2^n &\geq n^2 - 2n + 1 + 2 \\ &= n^2 - 2n + 3. \end{aligned} \tag{4.2}$$

From this we want to draw a contradiction, presumably a contradiction to $2^{n+1} \not\geq n^2 + 2$.

To get the contradiction, we want to convert the left-hand side of Equation 4.2 to 2^{n+1}. For this purpose, we multiply both sides by 2, giving

$$\begin{aligned} 2^{n+1} &= 2 \cdot 2^n \\ &\geq 2n^2 - 4n + 6\ . \end{aligned}$$

You may have gotten this far and wondered "What next?" Since we want to get a contradiction, we want to see a way to convert the right hand side into something like $n^2 + 2$. Thus we write

$$\begin{aligned} 2^{n+1} &\geq 2n^2 - 4n + 6 \\ &= n^2 + 2 + n^2 - 4n + 4 \\ &= n^2 + 2 + (n-2)^2 \\ &\geq n^2 + 2\ , \end{aligned} \tag{4.3}$$

since $(n-2)^2 \geq 0$. This is a contradiction, so there must not have been a smallest counter-example, so there must be no counter-example, and thus $2^n \geq n^2 + 2$ for all nonnegative integers n.

What implication did we have to prove above? Let us use $p(n)$ to stand for $2^{n+1} \geq n^2 + 2$. Then in Equations 4.2 and 4.3 we proved that $p(n-1) \Rightarrow p(n)$. Notice that at one point in our proof we had to note that we had considered the case with $n = 0$ already. This might lead you to ask whether it would make more sense to forget about the contradiction now that we have $p(n-1) \Rightarrow p(n)$ in hand and just observe that $p(0)$ and $p(n-1) \Rightarrow p(n)$ give us $p(1)$, $p(1)$ and $p(n-1) \Rightarrow p(n)$ give us $p(2)$, and so on so that we have $p(k)$ for every k without having to deal with any contradictions. This is an excellent question which we will address shortly.

Now let's consider a more complicated example in which $p(0)$ (and $p(1)$) are not true, but $p(n)$ is true for larger n. Notice that $2^{n+1} \not> n^2 + 3$ for $n = 0$ and 1, but $2^{n+1} > n^2 + 3$ for any larger n we look at at. Let us try to prove that $2^{n+1} > n^2 + 3$ for $n \geq 2$. So let us let $p'(n)$ be the statement $2^{n+1} > n^2 + 3$. We can easily prove $p'(2)$: since $8 = 2^3 \geq 2^2 + 3 = 7$. If you look back at your proof that $p(n-1) \Rightarrow p(n)$, you will see that, when $n \geq 2$, essentially the same proof applies to p' as well, that is $p'(n-1) \Rightarrow p'(n)$. Thus we conclude from $p'(2)$ and $p'(2) \Rightarrow p'(3)$ that $p'(3)$ is true, and similarly for $p'(4)$, and so on.

The Principle of Mathematical Induction

It may seem clear that this procedure of repeatedly using the implication will prove $p(n)$ (or $p'(n)$) for all n (or all $n \geq 2$). That observation is the central idea of the Principle of Mathematical

Induction, described in what follows. In a theoretical discussion of how one constructs the integers from first principles, this principle (or the equivalent principle that every set of nonnegative integers has a smallest element, thus letting us use the "smallest counter-example" technique) is one of the first principles we assume. Thus it is not surprising that the principle is a fundamental one in discrete mathematics. The principle of mathematical induction is usually described in two forms. The one we have talked about so far is called the "weak form."

The Weak Principle of Mathematical Induction. If the statement $p(b)$ is true, and the statement $p(n-1) \Rightarrow p(n)$ is true for all $n > b$, then $p(n)$ is true for all integers $n \geq b$.

Suppose, for example, we wish to give a direct inductive proof that $2^{n+1} > n^2 + 3$ for $n \geq 2$. We would proceed as follows. (The material in square brackets is not part of the proof; it is a running commentary on what is going on in the proof.)

> We shall prove by induction that $2^{n+1} > n^2+3$ for $n \geq 2$. First, $2^{2+1} = 2^3 = 8$, while $2^2 + 3 = 7$. [We just proved $p'(2)$. We will now proceed to prove $p'(n-1) \Rightarrow p'(n)$.] Suppose now that $n > 2$ and that $2^n > (n-1)^2 + 3$. [We just made the hypothesis of $p'(n-1)$ in order to use Rule 8 of our rules of inference.] Now multiply both sides of this inequality by 2, giving us
>
> $$\begin{aligned} 2^{n+1} &> 2(n^2 - 2n + 1) + 6 \\ &= n^2 + 3 + n^2 - 4n + 4 + 1 \\ &= n^2 + 3 + (n-2)^2 + 1 \ . \end{aligned}$$
>
> Since $(n-2)^2 + 1$ is positive for $n > 2$, this proves $2^{n+1} > n^2 + 3$. [We just showed that from the hypothesis of $p'(n-1)$ we can derive $p'(n)$. Now we can apply Rule 8 to assert that $p'(n-1) \Rightarrow p'(n)$.] Therefore
>
> $$2^n > (n-1)^2 + 3 \quad \Rightarrow \quad 2^{n+1} > n^2 + 3.$$
>
> Therefore by the principle of mathematical induction, $2^{n+1} > n^2 + 3$ for $n \geq 2$.

In the proof we just gave, the sentence "First, $2^{2+1} = 2^3 = 8$, while $2^2 + 3 = 7$" is called the *base case.* It consisted of proving that $p(b)$ is true, where in this case b is 2 and $p(n)$ is $2^{n+1} > n^2 + 3$. The sentence "Suppose now that $n > 2$ and that $2^n > (n-1)^2 + 3$." is called the *inductive hypothesis.* This is the assumption that $p(n-1)$ is true. In inductive proofs, we always make such a hypothesis in order to prove the implication $p(n-1) \Rightarrow p(n)$. The proof of the implication is called the *inductive step* of the proof. The final sentence of the proof is called the *inductive conclusion.*

Exercise 4.1-4 Use mathematical induction to show that

$$1 + 3 + 5 + \cdots + (2k-1) = k^2$$

for each positive integer k

Exercise 4.1-5 For what values of n is $2^n > n^2$? Use mathematical induction to show that your answer is correct.

For Exercise 4.1-4, we note that the formula holds when $k = 1$. Assume inductively that the forumla holds when $k = n - 1$, so that $1 + 3 + \cdots + (2n - 3) = (n - 1)^2$. Adding $2n - 1$ to both sides of this equation gives

$$\begin{aligned} 1 + 3 + \cdots + (2n - 3) + (2n - 1) &= n^2 - 2n + 1 + 2n - 1 \\ &= n^2 \,. \end{aligned} \tag{4.4}$$

Thus the formula holds when $k = n$, and so by the principle of mathematical induction, the formula holds for all positive integers k.

Notice that in our discussion of Exercise 4.1-4 we nowhere mentioned a statement $p(n)$. In fact, $p(n)$ is the statement we get by substituting n for k in the formula, and in Equation 4.4 we were proving $p(n - 1) \Rightarrow p(n)$. Next notice that we did not explicitly say we were going to give a proof by induction; instead we told the reader when we were making the inductive hypothesis by saying "Assume inductively that ...." This makes the prose flow nicely but still tells the reader that he or she is reading a proof by induction. Notice also how the notation in the statement of the Exercise helped us write the proof. If we state what we are trying to prove in terms of a variable other than n, say k, then we can assume that our desired statement holds when this variable (k) is $n - 1$ and then prove that the statement holds when $k = n$. Without this notational device, we have to either mention our statement $p(n)$ explicitly, or avoid any talking about substituting values into the formula we are trying to prove. Our proof above that $2^{n+1} > n^2 + 3$ demonstrates this last approach to writing an inductive proof in plain English. This is usually the "slickest" way of writing an inductive proof, but it is often the hardest to master. We will use this approach first for the next exercise.

For Exercise 4.1-5 we note that $2 > 1$, but then the inequality fails for $n = 2, 3, 4$. However, $32 > 25$. Now we assume inductively that for $n > 5$ we have $2^{n-1} > (n - 1)^2$. Multiplying by 2 gives us

$$2^n > 2(n^2 - 2n + 1) = n^2 + n^2 - 4n + 2 > n^2 + n^2 - n \cdot n = n^2,$$

since $n > 6$ implies that $-4n > -n \cdot n$. Thus by the principle of mathematical induction, $2^n > n^2$ for all $n \geq 5$.

Alternatively, we could write the following. Let $p(n)$ denote the inequality $2^n > n^2$. Then $p(5)$ is true because $32 > 25$. Assume that $p(n - 1)$ is true. This gives us $2^{n-1} > (n - 1)^2$. Multiplying by 2 gives

$$2^n > 2(n^2 - 2n + 1) = n^2 + n^2 - 4n + 2 > n^2 + n^2 - n \cdot n = n^2,$$

since $n > 6$ implies that $-4n > -n \cdot n$. Therefore $p(n - 1) \Rightarrow p(n)$. Thus by the principle of mathematical induction, $2^n > n^2$ for all $n \geq 5$.

Notice how the "slick" method simply assumes that the reader knows we are doing a proof by indcution from the context and mentally supplies the appropriate $p(n)$ and observes that we have proved $p(n - 1) \Rightarrow p(n)$ at the right moment.

Here is a slight variation of the technique of changing variables. To prove that $2^n > n^2$ when $n \geq 5$, we observe that the inequality holds when $n = 5$ since $32 > 25$. Assume inductively that the inequality holds when $n = k$, so that $2^k > k^2$. Now when $k \geq 5$, multiplying both sides of this inequality by 2 gives us

$$2^{k+1} > 2k^2 = k^2 + k^2 \geq k^2 + 5k > k^2 + 2k + 1 = (k + 1)^2,$$

since $k \geq 5$ implies that $k^2 \geq 5k$ and $5k = 2k+3k > 2k+1$. Thus by the principle of mathematical induction, $2^n > n^2$ for all $n \geq 5$.

This last variation of the proof illustrates two ideas. First, there is no need to save the name n for the variable we use in applying mathematical induction. We used k as our "inductive variable" in this case. Second, there is no need to restrict ourselves to proving the implication $p(n-1) \Rightarrow p(n)$. In this case, we proved the implication $p(k) \Rightarrow p(k+1)$, and clearly these two implications are equivalent as n ranges over all integers larger than b and as k ranges over all integers larger than or equal to b.

Strong Induction

In our proof of Euclid's division theorem we had a statement of the form $p(m,n)$ and, assuming that it was false, we chose a smallest m such that $p(m,n)$ is false for some n. This meant we could assume that $p(m',n)$ is true for **all** $m' < m$, and we needed this assumption, because we ended up showing that $p(m-n,n) \Rightarrow p(m,n)$ in order to get our contradiction. This situation is different from the examples we used to introduce mathematical induction, for in those we used an implication of the form $p(n-1) \Rightarrow p(n)$. The essence of our method in this case is that we have a statement $q(k)$ we want to prove, we suppose it is false so there is a smallest k for which $q(k)$ is false, meaning we may assume $q(k')$ is true for all k' in the universe of q with $k' < k$. We then use this assumption to derive a proof of $q(k)$, thus generating our contradiction. Again, we can avoid the step of generating a contradiction in the following way. Suppose first we have a proof of $q(0)$. Suppose also that we have a proof that

$$q(0) \wedge q(1) \wedge q(2) \wedge \ldots \wedge q(k-1) \Rightarrow q(k)$$

for all k larger than 0. Then from $q(0)$ we can prove $q(1)$, from $q(0) \wedge q(1)$ we can prove $q(2)$, from $q(0) \wedge q(1) \wedge q(2)$ we can prove $q(3)$ and so on giving us a proof of $q(n)$ for any n we desire. This is the other form of the mathematical induction principle; we use it when, as in Euclid's division theorem, we can get an implication of the form $q(k') \Rightarrow q(k)$ for *some* $k' < k$ or when we can get an implication of the form $q(0) \wedge q(1) \wedge q(2) \wedge \ldots \wedge q(k-1) \Rightarrow q(k)$. (As is the case in Euclid's division theorem, we often don't really know what the k' is, so in these cases the first kind of situation is really just a special case of the second. This is common in using this version of induction, so we do not treat the first of the two implications separately.) This leads us to the method of proof known as the Strong Principle of Mathematical Induction.

The Strong Principle of Mathematical Induction. If the statement $p(b)$ is true, and the statement $p(b) \wedge p(b+1) \wedge \ldots \wedge p(n-1) \Rightarrow p(n)$ is true for all $n > b$, then $p(n)$ is true for all integers $n \geq b$.

Exercise 4.1-6 Prove that every positive integer is a power of a prime number or the product of powers of prime numbers.

In this exercise we can observe that 1 is a power of a prime number; for example $1 = 2^0$. Suppose now we know that every number less than n is a power of a prime number or a product of powers of prime numbers. Then if n is not a prime number, it is a product of two smaller numbers, each of which is, by our supposition, a power of a prime number or a product of powers of prime numbers. Therefore n is a power of a prime number or a product of powers of prime numbers. Thus, by the strong principle of mathematical induction, every positive integer is a power of a prime number or a product of powers of prime numbers.

Note that there was no explicit mention of an implication of the form

$$p(b) \wedge p(b+1) \wedge \ldots \wedge p(n-1) \Rightarrow p(n)$$

. This is common with inductive proofs. Note also that we did not explicitly identify the base case or the inductive hypothesis in our proof. This is common too. Readers of inductive proofs are expected to recognize when the base case is being given and when an implication of the form $p(n-1) \Rightarrow p(n)$ or $p(b) \wedge p(b+1) \wedge \cdots \wedge p(n-1) \Rightarrow p(n)$ is being proved.

Mathematical induction is used frequently in discrete math and computer science. Many quantities that we are interested in measuring, such as running time, space, or output of a program, typically are restricted to positive integers, and thus mathematical induction is a natural way to prove facts about these quantities. We will use it frequently throughout this course. We typically will not distinguish between strong and weak induction, we just think of them both as induction. (In Exercise 4.1-14 and Exercise 4.1-15 at the end of the section you will be asked to derive each version of the principle from the other.)

Induction in general

To summarize what we have said so far, a proof by mathematical induction showing that a statement $p(n)$ is true for all integers $n \geq b$ consists of two steps. First we show that $p(b)$ is true. This is called "establishing a *base case*." Then we show either that for all $n > b$, $p(n-1) \Rightarrow p(n)$, or that for all $n > b$, $p(b) \wedge p(b+1) \wedge \ldots \wedge p(n-1) \Rightarrow p(n)$. For this purpose, we make either the *inductive hypothesis* of $p(n-1)$ or the *inductive hypothesis* $p(b) \wedge p(b+1) \wedge \ldots \wedge p(n-1)$. Then we derive $p(n)$ to complete the proof of the implication we desire, either $p(n-1 \Rightarrow p(n)$ or $p(b) \wedge p(b+1) \wedge \ldots \wedge p(n-1) \Rightarrow p(n)$. This is where the core of an inductive proof lies, and is usually where we need the most insight into what we are trying to prove. As we have discussed, this suffices to show that $p(n)$ is true for all $n \geq b$. You will often see inductive proofs in which the implication being proved is $p(n) \Rightarrow p(n+1)$. In light of our discussion of Exercise 4.1-5, it should be clear that this is simply another variation of writing an inductive proof.

It is important to realize that induction arises in many circumstances that do not fit the "pat" description we gave above. However, inductive proofs always involve two things. First we always need a base case or cases. Second, we need to show an implication that demonstrates that $p(n)$ is true given that $p(n')$ is true for some set of $n' < n$, or possibly we may need to show a set of such implications. For example, consider the problem of proving the following statement:

$$\sum_{i=0}^{n} \lfloor \frac{i}{2} \rfloor = \begin{cases} \frac{n^2}{4} & \text{if } n \text{ is even} \\ \frac{n^2-1}{4} & \text{if } n \text{ is odd} \end{cases} \tag{4.5}$$

In order to prove this, one must show that $p(0)$ is true, $p(1)$ is true, $p(n-2) \Rightarrow p(n)$ if n is odd, and that $p(n-2) \Rightarrow p(n)$, if n is even. Putting all these together, we see that our formulas hold for all n. We can view this as either two proofs by induction, one for even and one for odd numbers, or one proof in which we have two base cases and two methods of deriving results from previous ones. This second view is more profitable, because it expands our view of what induction means, and makes it easier to find inductive proofs. In particular we could find situations where we have just one implication to prove but several base cases to check to cover all cases, or just one base case, but several different implications to prove to cover all cases.

We can even consider more involved examples. Consider proving the following by induction:

$$\forall n \geq 1 (n \geq 2^{\lfloor \log_2 n \rfloor}). \tag{4.6}$$

While there may be easier ways to prove this statement, we will outline how a proof by induction could go. We could easily prove $p(1)$. We could then prove that, if n is a power of 2, that $p(n/2) \Rightarrow p(n)$. Finally we could show that, for n not a power of 2 and $a < n$, that $p(n) \Rightarrow p(n+a)$. These three things together would "cover" all the integers and would suffice to prove (4.6).

Logically speaking, we could rework the examples above so that they fit the pattern of strong induction. For example, when we prove a second base case, then we have just proved that the first base case implies it, because a true statement implies a true statement. Writing a description of mathematical induction that covers all kinds of base cases and implications one might want to consider in practice would simply give students one more unnecessary thing to memorize, so we shall not do so. However, in both the mathematics and the computer science literature, inductive proofs are written with multiple base cases and multiple implications with no effort to reduce them to one of the standard forms of mathematical induction. So long as it is possible to "cover" all the cases under consideration with such a proof, it can be rewritten as a standard inductive proof. Since readers of such proofs are expected to know this is possible, and since it adds unnecessary verbiage to a proof to do so, this is almost always left out.

Important Concepts, Formulas, and Theorems

1. *Weak Principle of Mathematical Induction.* The weak principle of mathematical induction states that

   > If the statement $p(b)$ is true, and the statement $p(n-1) \Rightarrow p(n)$ is true for all $n > b$, then $p(n)$ is true for all integers $n \geq b$.

2. *Strong Principle of Mathematical Induction.* The strong principle of mathematical induction states that

   > If the statement $p(b)$ is true, and the statement $p(b) \wedge p(b+1) \wedge \ldots \wedge p(n-1) \Rightarrow p(n)$ is true for all $n > b$, then $p(n)$ is true for all integers $n \geq b$.

3. *Base Case.* Every proof by mathematical induction, strong or weak, begins with a *base case* which establishes the result being proved for at least one value of the variable on which we are inducting. This base case should prove the result for the smallest value of the variable for which we are asserting the result. In a proof with multiple base cases, the base cases should cover all values of the variable which are not covered by the inductive step of the proof.

4. *Inductive Hypothesis.* Every proof by induction includes an inductive hypothesis in which we assume the result $p(n)$ we are trying to prove is true when $n = k - 1$ or when $n < k$ (or in which we assume an equivalent statement).

5. *Inductive Step.* Every proof by induction includes an inductive step in which we prove the implication that $p(k-1) \Rightarrow p(k)$ or the implication that $p(b) \wedge p(b+1) \wedge \cdots \wedge p(k-1) \Rightarrow p(k)$, or some equivalent implication.

6. *Inductive Conclusion.* A proof by mathematical induction should include, at least implicitly, a concluding statement of the form "Thus by the principle of mathematical induction ...," which asserts that by the principle of mathematical induction the result $p(n)$ which we are trying to prove is true for all values of n including and beyond the base case(s).

Problems

1. This exercise explores ways to prove that $\frac{2}{3}+\frac{2}{9}+\cdots+\frac{2}{3^n}=1-\left(\frac{1}{3}\right)^n$ for all positive integers n.

   (a) First, try proving the formula by contradiction. Thus you assume that there is some integer n that makes the formula false. Then there must be some smallest n that makes the formula false. Can this smallest n be 1? What do we know about $\frac{2}{3}+\frac{2}{9}+\cdots+\frac{2}{3^i}$ when i is a positive integer smaller than this smallest n? Is $n-1$ a positive integer for this smallest n? What do we know about $\frac{2}{3}+\frac{2}{9}+\cdots+\frac{2}{3^{n-1}}$ for this smallest n? Write this as an equation and add $\frac{2}{3^n}$ to both sides and simplify the right side. What does this say about our assumption that the formula is false? What can you conclude about the truth of the formula? If $p(k)$ is the statement $\frac{2}{3}+\frac{2}{9}+\cdots+\frac{2}{3^k}=1-\left(\frac{1}{3}\right)^k$, what implication did we prove in the process of deriving our contradiction?

   (b) What is the base step in a proof by mathematical induction that $\frac{2}{3}+\frac{2}{9}+\cdots+\frac{2}{3^n}=1-\left(\frac{1}{3}\right)^n$ for all positive integers n? What would you assume as an inductive hypothesis? What would you prove in the inductive step of a proof of this formula by induction? Prove it. What does the priciple of mathematical induction allow you to conclude? If $p(k)$ is the statement $\frac{2}{3}+\frac{2}{9}+\cdots+\frac{2}{3^k}=1-\left(\frac{1}{3}\right)^k$, what implication did we prove in the process of doing our proof by induction?

2. Use contradiction to prove that $1\cdot 2+2\cdot 3+\cdots+n(n+1)=\frac{n(n+1)(n+2)}{3}$.

3. Use induction to prove that $1\cdot 2+2\cdot 3+\cdots+n(n+1)=\frac{n(n+1)(n+2)}{3}$.

4. Prove that $1^3+2^3+3^3+\cdots+n^3=\frac{n^2(n+1)^2}{4}$.

5. Write a careful proof of Euclid's division theorem using strong induction.

6. Prove that $\sum_{i=j}^{n}\binom{i}{j}=\binom{n+1}{j+1}$. As well as the inductive proof that we are expecting, there is a nice "story" proof of this formula. It is well worth trying to figure it out.

7. Prove that every number greater than 7 is a sum of a nonnegative integer multiple of 3 and a nonnegative integer multiple of 5.

8. The usual definition of exponents in an advanced mathematics course (or an intermediate computer science course) is that $a^0=1$ and $a^{n+1}=a^n\cdot a$. Explain why this defines a^n for all nonnegative integers n. Prove the rule of exponents $a^{m+n}=a^m a^n$ from this definition.

9. Our arguments in favor of the sum principle were quite intuitive. In fact the sum principle for n sets follows from the sum principle for two sets. Use induction to prove the sum principle for a union of n sets from the sum principle for a union of two sets.

10. We have proved that every positive integer is a power of a prime number or a product of powers of prime numbers. Show that this factorization is unique in the following sense: If you have two factorizations of a positive integer, both factorizations use exactly the same primes, and each prime occurs to the same power in both factorizations. For this purpose, it is helpful to know that if a prime divides a product of integers, then it divides one of the integers in the product. (Another way to say this is that if a prime is a factor of a product of integers, then it is a factor of one of the integers in the product.)

11. Prove that $1^4 + 2^4 + \cdots + n^4 = O(n^5)$.

12. Find the error in the following "proof" that all positive integers n are equal. Let $p(n)$ be the statement that all numbers in an n-element set of positive integers are equal. Then $p(1)$ is true. Now assume $p(n-1)$ is true, and let N be the set of the first n integers. Let N' be the set of the first $n-1$ integers, and let N'' be the set of the last $n-1$ integers. Then by $p(n-1)$ all members of N' are equal and all members of N'' are equal. Thus the first $n-1$ elements of N are equal and the last $n-1$ elements of N are equal, and so all elements of N are equal. Thus all positive integers are equal.

13. Prove by induction that the number of subsets of an n-element set is 2^n.

14. Prove that the Strong Principal of Mathematical Induction implies the Weak Principal of Mathematical Induction.

15. Prove that the Weak Principal of Mathematical Induction implies the Strong Principal of Mathematical Induction.

16. Prove (4.5).

17. Prove (4.6) by induction.