4.2 Recursion, Recurrences and Induction

Recursion

Exercise 4.2-1 Describe the uses you have made of recursion in writing programs. Include as many as you can.

Exercise 4.2-2 Recall that in the Towers of Hanoi problem we have three pegs, and on one peg we have a stack of n disks, each smaller in diameter than the one below it. An allowable move consists of removing a disk from one peg and sliding it onto another peg so that it is not above another disk of smaller size. We are to determine how many allowable moves are needed to move the disks from one peg to another. Describe the strategy you used in a recursive program to solve this problem.

For the Tower of Hanoi problem, you told the computer that to solve the problem with no disks you do nothing. To solve the problem of moving all disks to peg 2, you deal with the problem of moving $n-1$ disks to peg 3, then move disk n to peg 2, and then deal with the problem of moving the $n-1$ disks on peg 3 to peg 2. Thus if $M(n)$ is the number of moves needed to move n disks from peg i to peg j, we have

$$M(n) = 2M(n-1) + 1.$$

This is an example of a **recurrence equation** or **recurrence**. A recurrence equation is one that tells us how to compute the nth term of a sequence from the $(n-1)$st term or some or all the preceding terms. To completely specify a function on the basis of a recurrence, we have to give enough information about the function to get started. This information is called the *initial condition (or the initial conditions)* for the recurrence. In this case we have said that $M(0) = 0$. Using this, we get from the recurrence that $M(1) = 1$, $M(2) = 3$, $M(3) = 7$, $M(4) = 15$, $M(5) = 31$, and are led to guess that $M(n) = 2^n - 1$.

Formally, we write our recurrence and initial condition together as

$$M(n) = \begin{cases} 0 & \text{if } n = 0 \\ 2M(n-1) + 1 & \text{otherwise} \end{cases} \tag{4.7}$$

Now we give an inductive proof that our guess is correct. The base case is trivial, as we have defined $M(0) = 0$, and $0 = 2^0 - 1$. For the inductive step, we assume that $n > 0$ and $M(n-1) = 2^{n-1} - 1$. From the recurrence, $M(n) = 2M(n-1) + 1$. But by the inductive hypothesis $M(n-1) = 2^{n-1} - 1$, so we get that:

$$\begin{aligned} M(n) &= 2M(n-1) + 1 && (4.8) \\ &= 2(2^{n-1} - 1) + 1 && (4.9) \\ &= 2^n - 1. && (4.10) \end{aligned}$$

The ease with which we solved this recurrence and proved our solution correct is no accident. Recursion, recurrences and induction are all intimately related. The relationship between recursion and recurrences is reasonably transparent, as recurrences give a natural way of analyzing recursive algorithms. They are both abstractions that allow you to specify the solution to an instance of a problem of size n as some function of solutions to smaller instances. Induction also falls naturally into this paradigm. Here, you are deriving a statement $p(n)$ from statements $p(n')$ for $n' < n$. Thus we really have three variations on the same theme.

We also observe, more concretely, that the mathematical correctness of solutions to recurrences is naturally proved via induction. In fact, the correctness of recurrences in describing the number of steps needed to solve a recursive problem is also naturally proved by induction. The recurrence or recursive structure of the problem makes it straightforward to set up the induction proof.

First order linear recurrences

Exercise 4.2-3 The empty set ($\emptyset$) is a set with no elements. How many subsets does it have? How many subsets does the one-element set $\{1\}$ have? How many subsets does the two-element $\{1, 2\}$ set have? How many of these contain 2? How many subsets does $\{1, 2, 3\}$ have? How many contain 3? Give a recurrence for the number $S(n)$ of subsets of an n-element set, and prove by induction that your recurrence is correct.

Exercise 4.2-4 When someone is paying off a loan with initial amount A and monthly payment M at an interest rate of p percent, the total amount $T(n)$ of the loan after n months is computed by adding $p/12$ percent to the amount due after $n-1$ months and then subtracting the monthly payment M. Convert this description into a recurrence for the amount owed after n months.

Exercise 4.2-5 Given the recurrence

$$T(n) = rT(n-1) + a,$$

where r and a are constants, find a recurrence that expresses $T(n)$ in terms of $T(n-2)$ instead of $T(n-1)$. Now find a recurrence that expresses $T(n)$ in terms of $T(n-3)$ instead of $T(n-2)$ or $T(n-1)$. Now find a recurrence that expresses $T(n)$ in terms of $T(n-4)$ rather than $T(n-1)$, $T(n-2)$, or $T(n-3)$. Based on your work so far, find a general formula for the solution to the recurrence

$$T(n) = rT(n-1) + a,$$

with $T(0) = b$, and where r and a are constants.

If we construct small examples for Exercise 4.2-3, we see that $\emptyset$ has only 1 subset, $\{1\}$ has 2 subsets, $\{1, 2\}$ has 4 subsets, and $\{1, 2, 3\}$ has 8 subsets. This gives us a good guess as to what the general formula is, but in order to prove it we will need to think recursively. Consider the subsets of $\{1, 2, 3\}$:

$$\begin{array}{cccc} \emptyset & \{1\} & \{2\} & \{1,2\} \\ \{3\} & \{1,3\} & \{2,3\} & \{1,2,3\} \end{array}$$

The first four subsets do not contain three, and the second four do. Further, the first four subsets are exactly the subsets of $\{1, 2\}$, while the second four are the four subsets of $\{1, 2\}$ with 3 added into each one. This suggests that the recurrence for the number of subsets of an n-element set (which we may assume is $\{1, 2, \ldots, n\}$) is

$$S(n) = \begin{cases} 2S(n-1) & \text{if } n \geq 1 \\ 1 & \text{if } n = 0 \end{cases}.$$

The main idea is that the subsets of an n-element set can be partitioned by whether they contain element n or not. The subsets of $\{1, 2, \ldots, n\}$ containing element n can be constructed by adjoining the element n to the subsets without element n. So the number of subsets with element n is the same as the number of subsets without element n. The number of subsets without element n is just the number of subsets of an $n-1$-element set. Thus the number of subsets of $\{1, 2, \ldots, n\}$ is twice the number of subsets of $\{1, 2, \ldots, n-1\}$. This will give us the inductive step of a proof by induction that the recurrence correctly describes the number of subsets of an n-element set for all n

How, exactly, does the inductive proof go? We know the initial condition of our recurrence properly describes the number of subsets of a zero-element set, namely 1 (remember that the empty set has itself and only itself as a subset). Now we suppose our recurrence properly describes the number of subsets of a k-element set for k less than n. Then since the number of subsets of an n-element set containing a specific element x is the same as the number of subsets without element x, the total number of number of subsets of an n-element set is just twice the number of subsets of an $(n-1)$-element set. Therefore, because the recurrence properly describes the number of subsets of an $(n-1)$-element subset, it properly describes the number of subsets of an n-element set. Therefore by the principle of mathematical induction, our recurrence describes the number of subsets of an n-element set for all integers $n \geq 0$. Notice how the proof by induction goes along exactly the same lines as our derivation of the recurrence. This is one of the beautiful facets of the relationship between recurrences and induction; usually what we do to figure out a recurrence is exactly what we would do to prove we are right!

For Exercise 4.2-4 we can algebraically describe what the problem said in words by

$$T(n) = (1 + .01p/12) \cdot T(n-1) - M,$$

with $T(0) = A$. Note that we add $.01p/12$ times the principal to the amount due each month, because $p/12$ percent of a number is $.01p/12$ times the number.

Iterating a recurrence

Turning to Exercise 4.2-5, we can substitute the right hand side of the equation $T(n-1) = rT(n-2) + a$ for $T(n-1)$ in our recurrence, and then substitute the similar equations for $T(n-2)$ and $T(n-3)$ to write

$$\begin{aligned}
T(n) &= r(rT(n-2) + a) + a \\
&= r^2 T(n-2) + ra + a \\
&= r^2(rT(n-3) + a) + ra + a \\
&= r^3 T(n-3) + r^2 a + ra + a \\
&= r^3(rT(n-4) + a) + r^2 a + ra + a \\
&= r^4 T(n-4) + r^3 a + r^2 a + ra + a
\end{aligned}$$

From this, we can guess that

$$\begin{aligned} T(n) &= r^n T(0) + a\sum_{i=0}^{n-1} r^i \\ &= r^n b + a\sum_{i=0}^{n-1} r^i. \end{aligned}$$

The method we used to guess the solution is called *iterating the recurrence*indexiteration of a recurrence because we repeatedly use the recurrence with smaller and smaller values in place of n. We could instead have written

$$\begin{aligned} T(0) &= b \\ T(1) &= rT(0) + a \\ &= rb + a \\ T(2) &= rT(1) + a \\ &= r(rb+a) + a \\ &= r^2 b + ra + a \\ T(3) &= rT(2) + a \\ &= r^3 b + r^2 a + ra + a \end{aligned}$$

This leads us to the same guess, so why have we introduced two methods? Having different approaches to solving a problem often yields insights we would not get with just one approach. For example, when we study recursion trees, we will see how to visualize the process of iterating certain kinds of recurrences in order to simplify the algebra involved in solving them.

Geometric series

You may recognize that sum $\sum_{i=0}^{n-1} r^i$. It is called a *finite geometric series with common ratio* r. The sum $\sum_{i=0}^{n-1} ar^i$ is called a *finite geometric series with common ratio r and initial value a.* Recall from algebra that $(1-x)(1+x) = 1-x^2$. You may or may not have seen in algebra that $(1-x)(1+x+x^2) = 1-x^3$ or $(1-x)(1+x+x^2+x^3) = 1-x^4$. These factorizations are easy to verify, and they suggest that $(1-r)(1+r+r^2+\cdots+r^{n-1}) = 1-r^n$, or

$$\sum_{i=0}^{n-1} r^i = \frac{1-r^n}{1-r}.$$

In fact this formula is true, and lets us rewrite the formula we got for $T(n)$ in a very nice form.

Theorem 4.1 *If $T(n) = rT(n-1) + a$, $T(0) = b$, and $r \neq 1$ then*

$$T(n) = r^n b + a\frac{1-r^n}{1-r}$$

for all nonnegative integers n.

Proof: We will prove our formula by induction. Notice that the formula gives $T(0) = r^0 b + a\frac{1-r^0}{1-r}$ which is b, so the formula is true when $n = 0$. Now assume that

$$T(n-1) = r^{n-1}b + a\frac{1-r^{n-1}}{1-r}.$$

Then we have

$$\begin{aligned} T(n) &= rT(n-1) + a \\ &= r\left(r^{n-1}b + a\frac{1-r^{n-1}}{1-r}\right) + a \\ &= r^n b + \frac{ar - ar^n}{1-r} + a \\ &= r^n b + \frac{ar - ar^n + a - ar}{1-r} \\ &= r^n b + a\frac{1-r^n}{1-r} \ . \end{aligned}$$

Therefore by the principle of mathematical induction, our formula holds for all integers n greater than 0. ■

Corollary 4.2 *The formula for the sum of a geometric series with $r \neq 1$ is*

$$\sum_{i=0}^{n-1} r^i = \frac{1-r^n}{1-r}.$$

Proof: Take $b = 0$ and $a = 1$ in Theorem 4.1. Then the sum $1 + r + \cdots + r^{n-1}$ satisfies the recurrence given. ■

Often, when we see a geometric series, we will only be concerned with expressing the sum in big-O notation. In this case, we can show that the sum of a geometric series is at most the largest term times a constant factor, where the constant factor depends on r, but not on n.

Lemma 4.3 *Let r be a quantity whose value is independent of n and not equal to 1. Let $t(n)$ be the largest term of the geometric series*

$$\sum_{i=0}^{n-1} r^i.$$

Then the value of the geometric series is $O(t(n))$.

Proof: It is straightforward to see that we may limit ourselves to proving the lemma for $r > 0$. We consider two cases, depending on whether $r > 1$ or $r < 1$. If $r > 1$, then

$$\begin{aligned} \sum_{i=0}^{n-1} r^i &= \frac{r^n - 1}{r-1} \\ &\leq \frac{r^n}{r-1} \\ &= r^{n-1}\frac{r}{r-1} \\ &= O(r^{n-1}). \end{aligned}$$

On the other hand, if $r < 1$, then the largest term is $r^0 = 1$, and the sum has value

$$\frac{1-r^n}{1-r} < \frac{1}{1-r}.$$

Thus the sum is $O(1)$, and since $t(n) = 1$, the sum is $O(t(n))$. ■

In fact, when r is nonnegative, an even stronger statement is true. Recall that we said that, for two functions f and g from the real numbers to the real numbers that $f = \Theta(g)$ if $f = O(g)$ and $g = O(f)$.

Theorem 4.4 *Let r be a nonnegative quantity whose value is independent of n and not equal to 1. Let $t(n)$ be the largest term of the geometric series*

$$\sum_{i=0}^{n-1} r^i.$$

Then the value of the geometric series is $\Theta(t(n))$.

Proof: By Lemma 4.3, we need only show that $t(n) = O(\frac{r^n-1}{r-1})$. Since all r^i are nonnegative, the sum $\sum_{i=0}^{n-1} r^i$ is at least as large as any of its summands. But $t(n)$ is one of these summands, so $t(n) = O(\frac{r^n-1}{r-1})$. ■

Note from the proof that $t(n)$ and the constant in the big-O upper bound depend on r. We will use this lemma in subsequent sections.

First order linear recurrences

A recurrence $T(n) = f(n)T(n-1) + g(n)$ is called a *first order linear recurrence*. When $f(n)$ is a constant, say r, the general solution is almost as easy to write down as in the case we already figured out. Iterating the recurrence gives us

$$\begin{aligned}
T(n) &= rT(n-1) + g(n) \\
&= r\Big(rT(n-2) + g(n-1)\Big) + g(n) \\
&= r^2T(n-2) + rg(n-1) + g(n) \\
&= r^2\Big(rT(n-3) + g(n-2)\Big) + rg(n-1) + g(n) \\
&= r^3T(n-3) + r^2g(n-2) + rg(n-1) + g(n) \\
&= r^3\Big(rT(n-4) + g(n-3)\Big) + r^2g(n-2) + rg(n-1) + g(n) \\
&= r^4T(n-4) + r^3g(n-3) + r^2g(n-2) + rg(n-1) + g(n) \\
&\vdots \\
&= r^nT(0) + \sum_{i=0}^{n-1} r^i g(n-i)
\end{aligned}$$

This suggests our next theorem.

Theorem 4.5 *For any positive constants a and r, and any function g defined on the nonnegative integers, the solution to the first order linear recurrence*

$$T(n) = \begin{cases} rT(n-1) + g(n) & \text{if } n > 0 \\ a & \text{if } n = 0 \end{cases}$$

is

$$T(n) = r^n a + \sum_{i=1}^{n} r^{n-i} g(i). \tag{4.11}$$

Proof: Let's prove this by induction.

Since the sum in equation 4.12 has no terms when $n = 0$, the formula gives $T(0) = 0$ and so is valid when $n = 0$. We now assume that n is positive and $T(n-1) = r^{n-1}a + \sum_{i=1}^{n-1} r^{(n-1)-i} g(i)$. Using the definition of the recurrence and the inductive hypothesis we get that

$$\begin{aligned} T(n) &= rT(n-1) + g(n) \\ &= r\left(r^{n-1}a + \sum_{i=1}^{n-1} r^{(n-1)-i} g(i)\right) + g(n) \\ &= r^n a + \sum_{i=1}^{n-1} r^{(n-1)+1-i} g(i) + g(n) \\ &= r^n a + \sum_{i=1}^{n-1} r^{n-i} g(i) + g(n) \\ &= r^n a + \sum_{i=1}^{n} r^{n-i} g(i). \end{aligned}$$

Therefore by the principle of mathematical induction, the solution to

$$T(n) = \begin{cases} rT(n-1) + g(n) & \text{if } n > 0 \\ a & \text{if } n = 0 \end{cases}$$

is given by Equation 4.12 for all nonnegative integers n. ■

The formula in Theorem 4.5 is a little less easy to use than that in Theorem 4.1, but for a number of commonly occurring functions g the sum $\sum_{i=1}^{n} r^{n-i} g(i)$ is reasonable to compute.

Exercise 4.2-6 Solve the recurrence $T(n) = 4T(n-1) + 2^n$ with $T(0) = 6$.

Using Equation 4.12, we can write

$$\begin{aligned} T(n) &= 6 \cdot 4^n + \sum_{i=1}^{n} 4^{n-i} \cdot 2^i \\ &= 6 \cdot 4^n + 4^n \sum_{i=1}^{n} 4^{-i} \cdot 2^i \\ &= 6 \cdot 4^n + 4^n \sum_{i=1}^{n} \left(\frac{1}{2}\right)^i \end{aligned}$$

$$
\begin{aligned}
&= 6 \cdot 4^n + 4^n \cdot \frac{1}{2} \cdot \sum_{i=0}^{n-1} \left(\frac{1}{2}\right)^i \\
&= 6 \cdot 4^n + (1 - (\frac{1}{2})^n) \cdot 4^n \\
&= 7 \cdot 4^n - 2^n
\end{aligned}
$$

Important Concepts, Formulas, and Theorems

1. *Recurrence Equation or Recurrence.* A recurrence equation is one that tells us how to compute the nth term of a sequence from the $(n-1)$st term or some or all the preceding terms.

2. *Initial Condition.* To completely specify a function on the basis of a recurrence, we have to give enough information about the function to get started. This information is called the *initial condition (or the initial conditions)* for the recurrence.

3. *First Order Linear Recurrence.* A recurrence $T(n) = f(n)T(n-1) + g(n)$ is called a *first order linear recurrence.*

4. *Constant Coefficient Recurrence.* A recurrence in which $T(n)$ is expressed in terms of a sum of constant multiples of $T(k)$ for certain values $k < n$ (and perhaps another function of n) is called a *constant coefficient recurrence.*

5. *Solution to a First Order Constant Coefficient Linear Recurrence.* If $T(n) = rT(n-1) + a$, $T(0) = b$, and $r \neq 1$ then
$$T(n) = r^n b + a\frac{1-r^n}{1-r}$$
for all nonnegative integers n.

6. *Finite Geometric Series.* A finite geometric series with common ratio r is a sum of the form $\sum_{i=0}^{n-1} r^i$. The formula for the sum of a geometric series with $r \neq 1$ is
$$\sum_{i=0}^{n-1} r^i = \frac{1-r^n}{1-r}.$$

7. *Big-Theta Bounds on the Sum of a Geometric Series.* Let r be a nonnegative quantity whose value is independent of n and not equal to 1. Let $t(n)$ be the largest term of the geometric series
$$\sum_{i=0}^{n-1} r^i.$$
Then the value of the geometric series is $\Theta(t(n))$.

8. *Solution to a First Order Linear Recurrence.* For any positive constants a and r, and any function g defined on the nonnegative integers, the solution to the first order linear recurrence
$$T(n) = \begin{cases} rT(n-1) + g(n) & \text{if } n > 0 \\ a & \text{if } n = 0 \end{cases}$$

is

$$T(n) = r^n a + \sum_{i=1}^{n} r^{n-i} g(i). \tag{4.12}$$

9. Iterating a Recurrence. We say we are *iterating* a recurrence when we guess its solution by using the equation that expresses $T(n)$ in terms of $T(k)$ for k smaller than n to re-express $T(n)$ in terms of $T(k)$ for k smaller than $n-1$, then for k smaller than $n-2$, and so on until we can guess the formula for the sum.

Problems

1. Solve the recurrence $M(n) = 2M(n-1) + 2$, with a base case of $M(1) = 1$. How does it differ from the solution to Equation 4.7?

2. Solve the recurrence $M(n) = 3M(n-1) + 1$, with a base case of $M(1) = 1$. How does it differ from the solution to Equation 4.7.

3. Solve the recurrence $M(n) = M(n-1) + 2$, with a base case of $M(1) = 1$. How does it differ from the solution to Equation 4.7. Using iteration of the recurrence, we can write

$$\begin{aligned} M(n) &= M(n-1) + 2 \\ &= M(n-2) + 2 + 2 \\ &= M(n-3) + 2 + 2 + 2 \\ &\vdots \\ &= M(n-i) + \underbrace{2 + 2 + \cdots + 2}_{i \text{ times}} \\ &= M(n-i) + 2i \end{aligned}$$

Since we are given that $M(1)$ is 1, we take $i = n-1$ to get

$$\begin{aligned} &= M(1) + 2(n-1) \\ &= 1 + 2(n-1) \\ &= 2n - 1 \end{aligned}$$

Now, we verify the correctness of our derivation through mathematical induction.

(a) Base Case : $n = 1$: $M(1) = 2 \cdot 1 - 1 = 1$. Thus, base case is true.

(b) Suppose inductively that $M(n-1) = 2(n-1) - 1$.
Now, for $M(n)$, we have

$$\begin{aligned} M(n) &= M(n-1) + 2 \\ &= 2n - 2 - 1 + 2 \\ &= 2n - 1 \end{aligned}$$

So, from $1, 2$ and the principle of mathematical induction, the statment is true for all n. We could have gotten the same result by using Theorem 4.1.

Here, the recurrence is of the form $M(n) = M(n-1) + 2$ while in (4.8), it is of the form $M(n) = 2M(n-1)+1$. Thus, in Theorem 4.1, r would have been 1, and thus our geometric series sums to a multiple of the number of terms.

4. There are m functions from a one-element set to the set $\{1, 2, \ldots, m\}$. How many functions are there from a two-element set to $\{1, 2, \ldots, m\}$? From a three-element set? Give a recurrence for the number $T(n)$ of functions from an n-element set to $\{1, 2, \ldots, m\}$. Solve the recurrence.

5. Solve the recurrence that you derived in Exercise 4.2-4.

6. At the end of each year, a state fish hatchery puts 2000 fish into a lake. The number of fish in the lake at the beginning of the year doubles due to reproduction by the end of the year. Give a recurrence for the number of fish in the lake after n years and solve the recurrence.

7. Consider the recurrence $T(n) = 3T(n-1) + 1$ with the initial condition that $T(0) = 2$. We know from Theorem 4.1 exactly what the solution is. Instead of using the theorem, try to guess the solution from the first four values of $T(n)$ and then try to guess the solution by iterating the recurrence four times.

8. What sort of big-O bound can we give on the value of a geometric series $1 + r + r^2 + \cdots + r^n$ with common ratio $r = 1$?

9. Solve the recurrence $T(n) = 2T(n-1) + n2^n$ with the initial condition that $T(0) = 1$.

10. Solve the recurrence $T(n) = 2T(n-1) + n^3 2^n$ with the initial condition that $T(0) = 2$.

11. Solve the recurrence $T(n) = 2T(n-1) + 3^n$ with $T(0) = 1$.

12. Solve the recurrence $T(n) = rT(n-1) + r^n$ with $T(0) = 1$.

13. Solve the recurrence $T(n) = rT(n-1) + r^{2n}$ with $T(0) = 1$

14. Solve the recurrence $T(n) = rT(n-1) + s^n$ with $T(0) = 1$.

15. Solve the recurrence $T(n) = rT(n-1) + n$ with $T(0) = 1$. (There is a sum here that you may not know; thinking about the derivative of $1 + x + x^2 + \cdots + x^n$ will help you figure it out.)

16. The Fibonacci numbers are defined by the recurrence

$$T(n) = \begin{cases} T(n-1) + T(n-2) & \text{if } n > 0 \\ 1 & \text{if } n = 0 \text{ or } n = 1 \end{cases}$$

   (a) Write down the first ten Fibonacci numbers.

   (b) Show that $(\frac{1+\sqrt{5}}{2})^n$ and $(\frac{1-\sqrt{5}}{2})^n$ are solutions to the equation $F(n) = F(n-1) + F(n-2)$.

(c) Why is

$$c_1(\frac{1+\sqrt{5}}{2})^n + c_2(\frac{1-\sqrt{5}}{2})^n$$

a solution to the equation $F(n) = F(n-1) + F(n-2)$ for any real numbers c_1 and c_2?

(d) Find constants c_1 and c_2 such that the Fibonacci numbers are given by

$$F(n) = c_1(\frac{1+\sqrt{5}}{2})^n + c_2(\frac{1-\sqrt{5}}{2})^n$$