

Mathematics 19
Introduction to Set Theory
Winter 2020

Instructor: M. Groszek
Distributive: QDS
Prerequisites: None

Around the beginning of the nineteenth century, efforts to place mathematics on an unassailable logical foundation collided with Russell's Paradox. This paradox was described by Russell in 1902.¹ It shows that the operation of set formation, gathering all objects sharing a given property into a set, can lead to fatal contradiction. Since set formation was being treated as intuitive and central to both logic and mathematics, this was an intellectual catastrophe. In fact, in 1931, Kurt Gödel's Incompleteness Theorems showed that a truly unassailable logical foundation for mathematics is impossible.²

Set theory, which had first revealed the problem, provided a solution. Georg Cantor began the development of modern set theory in the latter part of the eighteenth century. Cantor was aware of paradoxes in set theory (Russell's Paradox is really a variation of Cantor's Paradox), but was content to say that some collections of objects are "inconsistent multiplicities" that cannot be collected into a set.³ In 1908 Ernst Zermelo published a collection of axioms for set theory,⁴ which carefully enumerate what kinds of collections of objects can be formed into sets, and apparently avoid the paradoxes of Cantor, Russell, and others. ("Apparently" is the best we can do, as Gödel's Theorem means it is impossible to prove this.) The axiomatization of set theory allows us to place mathematics on a set-theoretical foundation that, if not entirely unassailable, has proved both defensible and lasting.

¹Russell, B., 1902. "Letter to Frege," in Jean van Heijenoort (ed.), *From Frege to Gödel*, Cambridge, Mass.: Harvard University Press, 1967, 124-125.

²Gödel, K., 1931, "Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I," Monatshefte für Mathematik Physik, 38: 173-198. English translation in Jean van Heijenoort (ed.), *From Frege to Gödel*, Cambridge, Mass.: Harvard University Press, 1967, 596-616

³Cantor, G., 1899. "Letter to Dedekind," in Jean van Heijenoort (ed.), *from Frege to Gödel*, Cambridge, Mass.: Harvard University Press, 1967, 124-125.

⁴Zermelo, E., 1908, "Untersuchungen über die Grundlagen der Mengenlehre I," Mathematische Annalen 65: 261-281. English translation in Jean van Heijenoort (ed.), *From Frege to Gödel*, Cambridge, Mass.: Harvard University Press, 1967, 596-616

Set theory is not only a foundation for mathematics, it is a beautiful and important subject in its own right. Mathematician David Hilbert famously described it as “the paradise that Cantor created for us.”⁵

This course takes the axiomatic approach advanced by Zermelo. We will learn the Zermelo-Fraenkel axioms for set theory and see how they can be justified. We will use these axioms to develop a highly structured picture of the set-theoretic universe. We will see how set theory becomes a foundation for mathematics by using sets to represent various kinds of mathematical objects. We will learn some techniques for proving things about sets, and we will begin to develop the skill of writing clear and logically correct mathematical arguments. Because set theory has been so widely adopted as a foundation for mathematics, pretty much all upper-level mathematics courses use some of the language and techniques of set theory, which makes Math 19 good preparation for further study of mathematics.

Learning objectives:

Students will be able to give reasons for taking an axiomatic approach to set theory.

Students will be able to use the language and basic notions of set theory in reading textbooks, solving problems, and writing proofs, in this and in subsequent courses.

Students will be able to write clear and correct proofs. This entails using, and communicating, valid logical reasoning.

Students will know the axioms of set theory, will be able to state them correctly (if informally), will be able to explain why they are justified based on the intuitive picture of the set theoretic universe being built up from below, and will be able to use them in solving problems and writing proofs. Writing proofs means both that students will be able to figure out how to prove things, at a level of difficulty appropriate to an introductory mathematics course, and will be able to communicate their arguments.

Students will know the important theorems and definitions, will be able to state them correctly, and will be able to use them in solving problems and writing proofs.

Textbook: *Elements of Set Theory*, Herbert B. Enderton,

⁵Hilbert, D., 1926, “Über das Unendliche”, *Mathematische Annalen*, 95: 161-190. Lecture given Münster, 4 June 1925. English translation in Jean van Heijenoort (ed.), *From Frege to Gödel*, Cambridge, Mass.: Harvard University Press, 1967, 367-392.

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Grading:
Exam 1: 50 points
Exam 2: 100 points
Final Exam: 100 points
Homework: 150 points
Total: 400 points

The Honor Principle

Academic integrity is at the core of our mission as mathematicians and educators, and we take it very seriously. We also believe in working and learning together.

The Honor Principle (Homework):

Collaboration is permitted and encouraged, but no copying, and to be clear, this means no copying even from a board or scrap of paper on which a solution was hashed out collaboratively. What a student turns in as a homework solution is to be his or her own understanding of how to do the problems. The solutions you submit must be written by you alone. Any copying (electronic or otherwise) of another person's solutions, in whole or in part, is a violation of the Academic Honor Code.

Moreover, if in working with someone they have provided you with an important idea or approach, they should be explicitly given credit in your writeup. Hints given in office hours need not be cited. Note: It is not necessarily sufficient to annotate your paper with a phrase like "I worked with Joe on all the problems." Individual ideas are to be credited at each instance; they represent intellectual property. Giving another person credit for an idea will never lower your grade. You get credit for your understanding no matter how you acquired it.

The Honor Principle (Exams):

On in-class exams, you may not receive assistance of any kind from any source (living, published, electronic, etc), except the professor, you may not

consult any written materials including your notes and textbook, and you may not give assistance to anyone.

On take-home exams, you may use your textbook, your own notes and homework, and any materials distributed in class, but you may not use outside sources, including but not limited to other textbooks and online sources. Matters of clarification are to be left to the professor. If you have any questions as to whether some action would be acceptable under the Academic Honor Code, please speak to me, and I will be glad to help clarify things. It is always easier to ask beforehand.

Disability Accommodations, Religious Observances, and Other Concerns

The following policies address some common concerns students may have, but certainly not all of them. If you have any questions or worries about the course or your participation in it, please talk to your instructor. We want all students to succeed.

Students with disabilities who may need disability-related academic adjustments and services for this course are encouraged to see their instructor privately as early in the term as possible. Students requiring disability-related academic adjustments and services must consult the Student Accessibility Services office (Carson Hall, Suite 125, , 646-9900, Student.Accessibility.Services@Dartmouth.edu).

Once SAS has authorized services, students must show the originally signed SAS Services and Consent Form and/or a letter on SAS letterhead to their instructor. As a first step, if you have questions about whether you qualify to receive academic adjustments and services, you should contact the SAS office. All inquiries and discussions will remain confidential.

The academic environment at Dartmouth is challenging, our terms are intensive, and classes are not the only demanding part of your life. There are a number of resources available to you on campus to support your wellness, including your undergraduate dean, Counseling and Human Development, and the Student Wellness Center .

The Sexual Respect Website at Dartmouth provides a wealth of information on your rights with regard to sexual respect and resources that are available to all in our community.

Please note that, as a faculty member, I am obligated to share disclosures regarding conduct under Title IX with Dartmouth's Title IX Coordinator. Confidential resources are also available, and include licensed medical or

counseling professionals (e.g., a licensed psychologist), staff members of organizations recognized as rape crisis centers under state law (such as WISE), and ordained clergy.

Should you have any questions, please feel free to contact Dartmouth's Title IX Coordinator or the Deputy Title IX Coordinator for the Guarini School. Their contact information can be found on the sexual respect website

Some students may wish to take part in religious observances that occur during this academic term. If you have a religious observance that conflicts with your participation in the course, please meet with your instructor before the end of the second week of the term to discuss appropriate accommodations.

It may be possible to accommodate conflicts with extracurricular activities, employment, or family responsibilities, although this is not guaranteed. Please meet with your instructor as soon as possible.