Exponential Functions

A Review of Exponents and Logarithms

In this lecture, we introduce the exponential functions, which is the third major type of function we will study in this class. Before we can study the exponential functions, we need to review the rules of exponentiation and taking logarithms.

Let a and b be two real numbers with $a > 0$. If b is a natural number, then we define a^b to be the number we get by multiplying a with itself b times. It turns out that we can define the number a^b when b is any real number. The value of a^b is not as important as the rules we use to evaluate exponents. They are:

- $a^b \cdot a^c = a^{b+c}$. (example: $2^{-2} \cdot 2^5 = 2^{-2+5} = 2^3 = 8$)
- $(a^b)^c = a^{bc}$. (example: $(27)^4 = (3^3)^4 = 3^{12}$)
- $a^0 = 1$. (example: $\pi^{5-5} = \pi^0 = 1$)
- $1^b = 1$. (example: $1^\pi = 1$.)
- $a^1 = a$. (example: $2.374^1 = 2.374$)
- $a^{-b} = \frac{1}{a^b}$. (example: $6^{-2} = \frac{1}{6^2} = \frac{1}{36}$)
- $a^{\frac{1}{b}} = \sqrt[b]{a}$. (example: $3^{\frac{1}{2}} = \sqrt{3}$)

You must become familiar with all of these rules if you are not already.

Now let a and b be two real numbers, both greater than zero, with $a \neq 1$. We define the number $\log_a b$, which is called the logarithm base a of b, to be the number c such that $b = a^c$. In other words,

$$a^{\log_a b} = b.$$

For example, $\log_2 8 = 3$, because $2^3 = 8$. It is very difficult to calculate most logarithms, so the most important part of understanding logarithms is knowing the rules of taking logarithms. Those rules are listed below:

- $\log_a(bc) = \log_a b + \log_a c$. (example: $\log_5 6 = \log_5(2 \cdot 3) = \log_5 2 + \log_5 3$.)
- $\log_a(b^c) = c \log_a b$. (example: $\log_7 9 = \log_7(3^2) = 2 \log_7 3$.)
- $\log_a 1 = 0$. (example: $\log_{4.379} 1 = 0$.)
- $\log_1 b$ is undefined.
- $\log_a a = 1$. (example: $\log_{0.04} 0.04 = 1$.)
- $\log_a \left(\frac{1}{b}\right) = -\log_a b$. (example: $\log_3 \left(\frac{1}{2}\right) = -\log_3 2$.)

Again, you need to learn all of these rules immediately if you do not already know them.

Finally, we note that there is a special number, $\mathrm{e} = 2.7182818284590...$, which plays an important role in exponential functions and in calculus in general. For a real number $b > 0$, we define the natural logarithm of b to be the logarithm base e of b:

$$\ln b = \log_{\mathrm{e}} b.$$

So, for example,

$$\ln(\mathrm{e}^2) = \log_{\mathrm{e}}(\mathrm{e}^2) = 2 \log_{\mathrm{e}} \mathrm{e} = 2 \cdot 1 = 2.$$

We will use this notation for natural logarithm when we discuss the derivatives of exponential functions later in this lecture.

Generalized Exponential Functions

We define the exponential function by the formula

$$f(x) = \mathrm{e}^x.$$

So the exponential function is the function we get by taking a real number x as the input and, as the output, getting e raised to the power of x.

For a real number $a > 0$, we define the generalized exponential function by the formula

$$f(x) = a^x.$$

So in this case, for every input x, we get back a raised to the power of x.

The graphs of the generalized exponential functions, including e^x, all have roughly the same shape. We plot the functions $f(x) = 2^x$ and $g(x) = 4^x$ using the numerical tables below:

x	2^x
-3	0.125
-2	0.25
-1	0.5
0	1
1	2
2	4
3	8

x	4^x
-3	0.015625
-2	0.0625
-1	0.25
0	1
1	4
2	16
3	64

First, we note that both graphs are always above the x-axis. This is because for all positive a and for all x, a^x is a positive number. Next we see that, for both graphs, the vertical intercept of the graph is 1. This is because one of the rules of exponentiation is that $a^0 = 1$ for all values of a. The slope of 4^x is greater than the slope of 2^x at $x = 0$, and at all values of x. For both graphs, when $x > 0$, the graph quickly rises, illustrating that as x increases, a^x becomes very large very quickly. As we would expect, the quantity 4^x rises faster than the quantity 2^x, so we have that for $x > 0$ the graph of the function $f(x) = 2^x$ is lower than the graph of the function $g(x) = 4^x$. When $x < 0$, both functions get smaller and smaller as x becomes more and more negative. In both cases, neither graph touches the x-axis, even though both functions are approaching zero as x becomes very negative. This is our first example of a horizontal asymptote, which is illustrated in a graph by the curve of that graph getting closer and closer to a horizontal line (in this case $y = 0$) as x becomes very positive or very negative. We will discuss horizontal asymptotes a little more below, but first we remark that the graph of $f(x) = 4^x$ is higher than that of $g(x) = 4^x$ when $x < 0$. This should make sense to you, and if it does not, you need to review the rules of exponentiation above to figure out why $4^x < 2^x$ when $x < 0$.

Now let us plot the function $h(x) = \left(\frac{1}{2}\right)^x$ using the numerical table below:

x	$\left(\frac{1}{2}\right)^x$
-3	8
-2	4
-1	2
0	1
1	0.5
2	0.25
3	0.125

We see that the graph of $h(x) = \left(\frac{1}{2}\right)^x$ is the mirror image along the y-axis of the graph of $f(x) = 2^x$. The graph of $h(x)$ is still always above the x-axis, and it intersects the y-axis at $(0, 1)$, but now it has a horizontal asymptote off to the right. This exemplifies a general phenomenon: when $a > 1$, the horizontal asymptote is off to the left, and when $0 < a < 1$, if is off to the right. Question: what happens to the shape of the graph of a^x when $a = 1$?

Properties of Generalized Exponential Functions

The major properties of generalized exponential functions $f(x) = a^x$, written algebraically, are listed below:

- **Positive Function:** If $f(x) = a^x$ for some positive real number a, then $f(x) > 0$ for all x. In other words, $f(x)$ is always positive, no matter what value of x we choose. As we stated above, this tells us that the graph of $f(x) = a^x$ will never touch the x-axis, although it will approach it (see the third property below).

- **Vertical Intercept:** We recall that the vertical intercept of a function is its value at $x = 0$, its height when it touches the y-axis. For generalized exponential functions we have for all real numbers $a > 0$ that
$$f(0) = a^0 = 1$$
by the rules of exponentiation. Therefore we get that the vertical intercept of all generalized exponential functions is equal to 1, which we write as $f(0) = 1$. This also tells us, as we saw when we plotted 2^x and 4^x, that the graphs of all generalized exponential functions pass through the point $(0, 1)$.

- **Horizontal Asymptote:** We stated above that, when $a > 1$, as x becomes more and more negative, the value of $f(x) = a^x$ becomes closer and closer to zero, but it always remains positive. One way to state the idea of x becoming more and more negative is to imagine x approaching negative infinity, which is written $x \to -\infty$. It is important to note that x can never be $-\infty$, since $-\infty$ is not a real number, but we can make x very, very negative, which is what we mean by x approaching negative infinity. As x approaches negative infinity, the value of $f(x)$ becomes very close to zero. What we mean here is that, no matter how close we want the value of $f(x)$ to be to zero, we can always find a sufficiently large value of x so that $f(x)$ is that close to zero. We write this property with the following algebraic notation:
$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} a^x = 0.$$
(We wrote the notation twice, first with $f(x)$, then with a^x, to emphasize that $f(x)$ is a generalized exponential function. You do not have to write the notation twice in practice.) We verbalize this notation by saying "the limit of $f(x)$ as x approaches negative infinity is zero." Another way to verbalize this is to say "a^x has a left horizontal asymptote at $y = 0$" when $a > 1$. In general, we say that a function $f(x)$ has a left horizontal asymptote at $y = b$ if
$$\lim_{x \to -\infty} f(x) = b,$$
In other words, as x gets more and more negative, the value of $f(x)$ gets closer and closer to b.

  Likewise, when $0 < a < 1$, as x becomes more and more positive, $f(x)$ gets closer and closer to zero. Thus we write that
$$\lim_{x \to \infty} f(x) = \lim_{x \to -\infty} a^x = 0$$
and say that $f(x)$ has a right horizontal asymptote at $y = 0$ when $0 < a < 1$. In general, we say that $f(x)$ has a right horizontal asymptote at $y = b$ if
$$\lim_{x \to \infty} f(x) = b.$$
This means that, as x gets more and more positive, the value of $f(x)$ gets closer and closer to b.

  Finally, we note that, when $a = 1$, we have that
$$f(x) = a^x = 1^x = 1,$$
so $f(x)$ is the constant function 1. Technically, $f(x)$ has both a left horizontal asymptote and a right horizontal asymptote at $y = 1$, but since the graph of $f(x)$ is the line $y = 1$, we usually do not talk about its asymptotes.

Differentiating Exponential Functions

Let $f(x)$ be a generalized exponential function, that is, let $f(x) = a^x$ for some positive real number a. Then the formula for the derivative of $f(x)$ is

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(a^x) = (\ln a)a^x.$$

So, to get the derivative of a^x, we simply multiply it by $\ln a$. In particular, we have that the derivative of the exponential function e^x is itself:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^x) = \mathrm{e}^x.$$

This is a very special property, because the only functions for which the derivative of the function equals itself are e^x and its constant multiples. If you continue on to higher calculus classes, you will find that the function e^x comes up all the time in the solutions of differential equations. This is also the main reason why the number e is so important in mathematics.

Let us do some examples of differentiating generalized exponential functions. First, let $f(x) = 7^x$. Then

$$f'(x) = (\ln 7)7^x.$$

There is a second way to get this result. First, we change the formula of $f(x)$ by noticing that $7 = \mathrm{e}^{\ln 7}$:

$$f(x) = 7^x = (\mathrm{e}^{\ln 7})^x = \mathrm{e}^{(\ln 7)x}.$$

This is a very useful trick to remember, and you should try to remember it. Next, we evaluate our new formula for $f(x)$ using the Chain Rule: the outer function is e^x, and the inner function is $(\ln 7)x$, so we get that

$$f'(x) = \mathrm{e}^{(\ln 7)x} \cdot (\ln 7) = (\ln 7)\mathrm{e}^{(\ln 7)x} = (\ln 7)7^x,$$

which is exactly what we got before. Notice that now, instead of remembering the formula for the derivative of all generalized exponential functions, all we have to know is that the derivative of e^x is itself, and how to apply the Chain Rule. How you choose to remember how to differentiate $f(x) = 7^x$, either by formula or by Chain Rule, is your choice.

Let us do a harder example of using the Chain Rule with exponential functions. Let $f(x) = \mathrm{e}^{2x^3-4x-7}$. The inner function is $2x^3 - 4x - 7$, and its derivative is $6x^2 - 4$. The outer function, the function applied last, is e^x, and its derivative is e^x. So the derivative of e^x at $2x^3 - 4x - 7$ is e^{2x^3-4x-7}. Putting all of this together with the Chain Rule, we get that

$$f'(x) = \mathrm{e}^{2x^3-4x-7} \cdot (6x^2 - 4) = (6x^2 - 4)\mathrm{e}^{2x^3-4x-7}.$$

Finally, let us take

$$f(x) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}x^2}.$$

This is the equation for the Normal Distribution, also known as the Bell Curve. Roughly speaking, the Bell Curve tells us the distribution of results for processes of chance. The uses of the function $f(x)$ are not the concerns of this class; what we want to be able to do is differentiate $f(x)$. First, we recognize that we can pull a constant multiple out of the derivative:

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}x^2}\right) = \frac{1}{\sqrt{2\pi}}\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-\frac{1}{2}x^2}\right).$$

Next, we use the Chain Rule: the outer function is e^x, and the inner function is $-\frac{1}{2}x^2$. The derivative of the inner function is therefore $-x$. Putting this all together using the Chain Rule, we get that

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{1}{\sqrt{2\pi}}\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-\frac{1}{2}x^2}\right) = \frac{1}{\sqrt{2\pi}} \cdot \mathrm{e}^{-\frac{1}{2}x^2} \cdot (-x) = -\frac{x}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}x^2}.$$

On your own, find the second derivative of $f(x)$, and find the critical points of $f(x)$ (there should be only one). Is the critical point a local maximum or a local minimum? Does your answer make sense given the shape of the Bell Curve?