Rational Polynomial Functions

Rational Polynomial Functions and Their Domains

Today we discuss rational polynomial functions. A function $f(x)$ is a rational polynomial function if it is the quotient of two polynomials $p(x)$ and $q(x)$:

$$f(x) = \frac{p(x)}{q(x)}.$$

Below we list three examples of rational polynomial functions:

$$f(x) = \frac{x^2 - 6x + 5}{x + 1} \qquad g(x) = \frac{x^2 - 9}{x + 3} \qquad h(x) = \frac{x + 3}{x^2 + 5x + 4}$$

We already know how to find the domains of rational polynomial functions, at least in principle: the domain of $f(x)$ is all real numbers x for which its denominator $q(x)$ is not equal to 0. Thus in our three examples above, the domains are, respectively:

$$\mathrm{Dom}(f) = \{x \in \mathbb{R} : x \neq -1\} \qquad \mathrm{Dom}(g) = \{x \in \mathbb{R} : x \neq -3\} \qquad \mathrm{Dom}(h) = \{x \in \mathbb{R} : x \neq -1, -4\}$$

We also know how to find the derivatives of rational polynomial functions on their domains using the quotient rule. Thus we have that

$$\begin{aligned}
f'(x) &= \frac{(x+1)\cdot(2x-6) - (x^2-6x+5)\cdot 1}{(x+1)^2} = \frac{2x^2-4x-6-x^2-6x+5}{x^2+2x+1} = \frac{x^2-10x-1}{x^2+2x+1} \\
g'(x) &= \frac{(x+3)\cdot 2x - (x^2-9)\cdot 1}{(x+3)^2} = \frac{2x^2+6x-x^2+9}{x^2+6x+9} = \frac{x^2+6x+9}{x^2+6x+9} = 1 \\
h'(x) &= \frac{(x^2+5x+4)\cdot 1 - (x+3)\cdot(2x+5)}{(x^2+5x+4)^2} = \frac{x^2+5x+4-2x^2-11x-15}{(x^2+5x+4)^2} = \frac{-x^2-6x-11}{(x^2+5x+4)^2}.
\end{aligned}$$

Our job today is to learn about the limit behavior of rational polynomial functions.

Polynomial Long Division

Our main tool for understanding the limit behavior of rational polynomial functions, that is, their behavior as x gets very positive or very negative, or as x approaches a point outside its domain, is polynomial long division. Polynomial long division is a way to reduce a rational polynomial function into the sum of a polynomial function and a much smaller rational polynomial function. The polynomial function in this sum will tell us how our original rational polynomial function behaves when x gets very positive or very negative; the smaller rational polynomial function will tell us how our original function behaves where around points outside its domain.

In order to understand polynomial long division, we will do several examples. First, take

$$f(x) = \frac{x^2 - 6x + 5}{x + 1}.$$

The leading term of the numerator (the term with the highest exponent) is x^2, and the leading term of the denominator is x. We ask the following question: by what do we multiply the leading term of the denominator to get the leading term of the numerator? In this case, the answer is x, because x times the leading term of the denominator, which is also x, gives us x^2, the leading term of the numerator. What we are now going to do is rewrite $f(x)$ as the sum of x, the answer to the question above, and some smaller rational polynomial function $g(x)$:

$$f(x) = \frac{x^2 - 6x + 5}{x + 1} = x + g(x).$$

We now want to find a formula for $g(x)$. We do this by subtracting x from both sides:

$$\frac{x^2-6x+5}{x+1} = x + g(x)$$
$$\frac{x^2-6x+5}{x+1} - x = g(x).$$

Now we want to simplify this formula for $g(x)$, and we do that by putting all of its terms under the same denominator:

$$\begin{aligned} g(x) &= \frac{x^2-6x+5}{x+1} - x = \frac{x^2-6x+5}{x+1} - x\frac{x+1}{x+1} = \frac{x^2-6x+5}{x+1} - \frac{x^2+x}{x+1} \\ &= \frac{x^2-6x+5-x^2-x}{x+1} = \frac{-7x+5}{x+1}. \end{aligned}$$

Thus we can write $f(x)$ as the sum of x and the smaller rational polynomial function $g(x)$:

$$f(x) = \frac{x^2-6x+5}{x+1} = x + \frac{-7x+5}{x+1}.$$

Of course, we can do the same to $g(x)$: the leading term of its numerator is $-7x$, and the leading term of its denominator is x, just as before. We ask the question: by what do we have to multiply the leading term of the denominator to get the leading term of the numerator? In this case, $-7x = -7 \cdot x$, so the answer to that question is -7. Thus we can rewrite $g(x)$ as the sum of -7 and some smaller rational polynomial function $h(x)$:

$$g(x) = \frac{-7x+5}{x+1} = -7 + h(x).$$

To get a formula for $h(x)$, we subtract -7 from both sides of the equation above:

$$\frac{-7x+5}{x+1} = -7 + h(x)$$
$$\frac{-7x+5}{x+1} + 7 = h(x)$$

Next, we simplify our formula for $h(x)$ by putting all of its terms under a common denominator:

$$h(x) = \frac{-7x+5}{x+1} + 7 = \frac{-7x+5}{x+1} + 7\frac{x+1}{x+1} = \frac{-7x+5}{x+1} + \frac{7x+7}{x+1} = \frac{-7x+5+7x+7}{x+1} = \frac{12}{x+1}.$$

Thus we have that

$$g(x) = \frac{-7x+5}{x+1} = -7 + \frac{12}{x+1}.$$

which then tells us a new formula for $f(x)$:

$$f(x) = \frac{x^2-6x+5}{x+1} = x + \frac{-7x+5}{x+1} = x - 7 + \frac{12}{x+1}.$$

This is where we stop, because the leading term of $h(x)$, which is 12, is too small to be divided into by x. In general, we stop the process we just described, the process of polynomial long division, when the leading term of the numerator has a smaller exponent than the leading term of the denominator, or simply when, in this case, the numerator is a constant. we will learn how to use this new formula for our original rational polynomial function $f(x)$ to understand the limit behavior of $f(x)$ in the next section, but first let us do another example of polynomial long division.

The next example from the previous section of this lecture is

$$g(x) = \frac{x^2-9}{x+3}.$$

We apply polynomial long division to $g(x)$: the leading term of the numerator is x^2, and the leading term of the denominator is x. We ask: by what must we multiply x to get x^2? The answer, just as before, is x, so we rewrite $g(x)$ as the sum of x and a smaller rational polynomial function $h(x)$:

$$g(x) = \frac{x^2 - 9}{x + 3} = x + h(x).$$

We next subtract x from both sides to get a formula for $h(x)$:

$$h(x) = \frac{x^2 - 9}{x + 3} - x.$$

Now we simplify that formula:

$$h(x) = \frac{x^2 - 9}{x + 3} - x = \frac{x^2 - 9}{x + 3} - x\frac{x + 3}{x + 3} = \frac{x^2 - 9}{x + 3} - \frac{x^2 + 3x}{x + 3} = \frac{x^2 - 9 - x^2 - 3x}{x + 3} = \frac{-3x - 9}{x + 3}.$$

Thus we have that

$$g(x) = \frac{x^2 - 9}{x + 3} = x + \frac{-3x - 9}{x + 3}.$$

Now we apply polynomial long division to $h(x)$: the leading term of the numerator is $-3x$, and the leading term of the denominator is x. We know that $-3x = -3 \cdot x$, so we rewrite $h(x)$ as the sum of -3 and some smaller rational polynomial function $k(x)$:

$$h(x) = \frac{-3x - 9}{x + 3} = -3 + k(x).$$

Solving for $k(x)$, we get

$$k(x) = \frac{-3x - 9}{x + 3} + 3.$$

We now simplify this formula for $k(x)$:

$$k(x) = \frac{-3x - 9}{x + 3} + 3 = \frac{-3x - 9}{x + 3} + 3\frac{x + 3}{x + 3} = \frac{-3x - 9}{x + 3} + \frac{3x + 9}{x + 3} = \frac{-3x - 9 + 3x + 9}{x + 3} = 0.$$

Hence $h(x) = -3$, so

$$g(x) = \frac{x^2 - 9}{x + 3} = x + h(x) = x - 3.$$

Actually, this is not precisely true: $g(x)$ equals $x - 3$ on its domain, which does not include -3. The function $g(x)$ and the function $x - 3$ are not the same function, because they have different domains. The graph of $x - 3$ is a line, and the graph of $g(x)$ is a line with a hole at $x = -3$. You need to be careful about this difference.

Finally, let us the third example from the previous section: we have that

$$h(x) = \frac{x + 3}{x^2 + 5x + 4}.$$

This is an example of a rational polynomial function to which we cannot apply polynomial long division, because the leading term of the numerator, which is x, has a smaller exponent than the leading term of the denominator, which is x^2. Thus we leave the formula of $h(x)$ alone for the purposes of studying its limit behavior.

Asymptotes and Limits of Rational Polynomial Functions

Given a rational polynomial function

$$f(x) = \frac{p(x)}{q(x)},$$

we can use polynomial long division to rewrite its formula so that it is the sum of some polynomial $a(x)$ and some rational polynomial function $r(x)$, the leading of the numerator of which has smaller exponent (or is a constant) than the leading term of its denominator:

$$f(x) = a(x) + r(x).$$

So, in our first example, we have that

$$f(x) = \frac{x^2 - 6x + 5}{x+1} = x + \frac{-7x+5}{x+1} = x - 7 + \frac{12}{x+1},$$

so $a(x) = x - 7$ and $r(x) = \frac{12}{x+1}$. In our second example, we got that

$$g(x) = \frac{x^2 - 9}{x+3} = x - 3,$$

so $a(x) = x - 3$, and since there is no quotient in this formula, $r(x) = 0$. For our third example, we have that

$$h(x) = \frac{x+3}{x^2+5x+4},$$

and we could not do polynomial long division, so $a(x) = 0$ and $r(x) = \frac{x+3}{x^2+5x+4}$. We now use $a(x)$ and $r(x)$ to understand the limit behavior of rational polynomial functions.

First, we note that, as x gets more and more positive, $r(x)$ gets closer and closer to 0. This is because the denominator, having a leading term of higher exponent than that of the numerator, will have a much greater absolute value than the numerator when x is a very large positive number. When we divide a number by another number that has a much bigger absolute value than it, we get a quotient which is very close to 0. Another way to say this is that $r(x)$ has a right vertical asymptote at $y = 0$. By the same reasoning, $r(x)$ also has a left vertical asymptote at $y = 0$, so that as x gets more and more negative, $r(x)$ gets closer and closer to 0.

In terms of $f(x) = a(x) + r(x)$, the horizontal asymptotes of $r(x)$ tell us that, when x is a very positive number or a very negative number, $r(x)$ is very close to 0, so $r(x)$ does not add much to $f(x)$ for very positive or very negative numbers. In other words, for very positive x or very negative x, the rational polynomial function $f(x)$ behaves basically like its polynomial part $a(x)$. So, in our first example, $f(x) = \frac{x^2-6x+5}{x+1}$ behaves basically like the linear function $x - 7$ when x is very positive or very negative. In our second example, $g(x) = \frac{x^2-9}{x+3}$ behaves like the linear function $x + 3$ for x of large absolute value. Finally, for our third example, $h(x) = \frac{x+3}{x^2+5x+4}$ behaves basically like the constant function 0 when x is very positive or very negative, since we said that $a(x) = 0$ in this case. In other words, $h(x)$ has both a left horizontal asymptote and a right horizontal asymptote at $y = 0$.

So, the behavior of a rational polynomial function $f(x)$ depends on its polynomial part $a(x)$. Specifically, we have that:

- if $a(x)$ is a constant function k, and in particular if $a(x) = 0$ (like for $h(x)$), then $f(x)$ has both a left horizontal asymptote at $y = k$ ana right horizontal asymptote at $y = k$.

- if $a(x)$ is a linear function $mx + b$, then as x gets very positive or very negative, the graph of $f(x)$ approaches the line $y = mx + b$, that is, it asymptotes to that line. We say then that $f(x)$ has both a left slant asymptote at $y = mx + b$ and a right slant asymptote at $y = mx + b$.

- if $a(x)$ is a higher polynomial, like a quadratic function or a cubic function, then there is no specific name for the behavior of $f(x)$ at very positive or very negative values of x, but the principle is the same: as x gets very positive or very negative, the graph of $f(x)$ asymptotes, that is, gets closer and closer, to the graph of $a(x)$. So, for example, if $a(x)$ is a quadratic function, $f(x)$ will look very much like a parabola when x is very positive or very negative.

Now let us consider the behavior of rational polynomial functions near values of x which are outside of their domains. The first thing we need to observe is that the remainder function $r(x)$, if it is not equal to

0 as in the case of the example $g(x)$ above, has the same denominator as the original rational polynomial function $f(x)$. This means that $r(x)$ and $f(x)$ have the same domain. Let b be a value of x at which $f(x)$ (and, therefore, $r(x)$) is undefined. As x approaches b from either the left or the right, the denominator of $r(x)$ will approach 0. Therefore, the denominator of $r(x)$ is guaranteed to become very small for values of x very close to b (what concept are we using here?). Meanwhile, the numerator of the remainder function $r(x)$ is almost certainly not approaching 0 as x approaches b (we will set aside the possibility that the numerator does approach 0 until the next class). This means that we are dividing a non-zero number by a very small number, so $r(x)$ will either become very positive or very negative as x approaches b.

Meanwhile, let us consider the behavior of the polynomial part $a(x)$ of the rational polynomial function $f(x)$ as x approaches b. Polynomial functions are continuous everywhere, so, as x approaches b from either the left or the right, $a(x)$ will approach $a(b)$, which is some finite number. So $f(x)$ near $x = b$ is the sum of a number near $a(b)$, a finite number, and the value of $r(x)$, which is either very positive or very negative. So, essentially, $f(x)$ will behave like $r(x)$ near points outside of its domain. If $r(x)$ has a positive left vertical asymptote at $x = b$, then so will $f(x)$; if $r(x)$ has a negative right vertical asymptote at $x = b$, then $f(x)$ will also have one; and so on. So, to understand the asymptotic behavior of $f(x)$ near points outside of its domain, we need to find the asymptotic behavior of $r(x)$ at those same points.

For example, in our first example, we said that the remainder function of $f(x) = \frac{x^2-6x+5}{x+1}$ is $r(x) = \frac{12}{x+1}$. We need to find the asymptotic behavior of $r(x)$ as x approaches -1, the one value of x outside of the domain of $f(x)$. The remainder function $r(x)$ is just a positive constant divided by $x+1$. When $x < -1$, $x+1$ is less than 0, so $r(x)$ will also be less than 0. Since the numerator of $r(x)$ is a constant, and $x + 1$ approaches 0 from both sides as x approaches -1, $r(x)$ has a left negative vertical asymptote at $x = -1$. Therefore, $f(x)$ also has a left negative vertical asymptote at $x = -1$:

$$\lim_{x \to -1^-} f(x) = -\infty.$$

Likewise, when $x > -1$, $x+1$ is greater than 0. This tells us that $r(x)$ has a right positive vertical asymptote at $x = -1$, and that $f(x)$ has one there as well:

$$\lim_{x \to -1^+} f(x) = +\infty.$$

This, together with the slant asymptotes of $f(x)$ that we found before, gives us the limit behavior of $f(x)$.

Now consider the third example $h(x) = \frac{x+3}{x^2+5x+4}$. We said that $h(x)$ is its own remainder, and that the points $x = -1$ and $x = -4$ are the only values of x outside of the domain of $h(x)$. As x approaches -1 from the left, the denominator of $h(x)$, which can be rewritten as $(x+1)(x+4)$, is negative, since $x+1$ is negative here and $x + 4$ are positive, while the numerator, $x + 3$, is positive. This means that $h(x)$ is negative as x approaches -1 from the left. The numerator $x + 3$ does not approach 0 at either of the points outside of the domain of $h(x)$, so $h(x)$ has a left positive negative asymptote as $x = -1$:

$$\lim_{x \to -1} h(x) = -\infty.$$

As x approaches -1 from the right side, the denominator is positive, since $x + 1$ is positive and $x + 4$ is positive, while the numerator is still positive. Thus $h(x)$ has a right positive vertical asymptote at $x = -1$:

$$\lim_{x \to -1^+} h(x) = +\infty.$$

We leave it to the reader to find the asymptotic behavior of $h(x)$ at $x = -4$ as an exercise.

Finally, for our second example above, $g(x) = \frac{x^2-9}{x+3}$, we found a remainder function $r(x) = 0$. This is a special case, since $r(x)$ is continuous everywhere, since it is a constant function. This means that it does not have any left or right vertical asymptotes at the points outside of the domain of $g(x)$. Since $a(x) = x - 3$ is also continuous at these points, $g(x)$ will not have vertical asymptotes at the one value of x outside of its domain, which is -3. Rather, $g(x)$ has left and right limits, and in fact simply a limit, at $x = -3$, which is exactly what we would expect:

$$\lim_{x \to -3} g(x) = -6.$$

Thus, when $r(x) = 0$, the original rational polynomial function has a limit at each of the points outside of its domain.