

# Constant and Linear Functions

## Constant Functions

In the first lecture, we introduced the notion of a real-valued function as a tool for investigating the real world. In today's lecture, we begin our study of the types of real-valued functions commonly used by scientists and engineers.

The simplest of all types of real-valued functions are the constant functions. A constant function is a function which takes the same value for  $f(x)$  no matter what  $x$  is. When we are talking about a generic constant function, we usually write  $f(x) = c$ , where  $c$  is some unspecified constant. Examples of constant functions include  $f(x) = 0$ ,  $f(x) = 1$ ,  $f(x) = \pi$ ,  $f(x) = -0.456238496$ , and  $f(x)$  equalling any other real number you can dream up. The great thing about constant functions is that you can plug in any real number you want for  $x$ , and you instantly know the value of the function at that  $x$ , no calculators needed. For example, suppose  $f(x) = 1$ . What is  $f(.28452041819)$ ? Say 1 and you're done.

The graph of a constant function is also very simple. Remember, the graph of a function  $f$  is a curve where every point  $(x, y)$  on the curve is such that  $y = f(x)$ . Suppose that we want to graph the function  $f(x) = 2$ . What do we know about the value of  $y$  for any point  $(x, y)$  on the graph of  $f(x)$ ? We know that if  $(x, y)$  is on the graph, then  $y = f(x) = 2$ , so every point on the graph of  $f(x) = 2$  is of the form  $(x, 2)$ . If we say then that the domain, the set of inputs, is the whole real line, this tells us that the graph of  $f(x) = 2$  is a horizontal line which passes through the  $y$ -axis at the point  $(0, 2)$ . In general, the graph of the function  $f(x) = c$  is the horizontal line which passes through the  $y$ -axis at the point  $(0, c)$ . In particular, the graph of the function  $f(x) = 0$  is the  $x$ -axis.

Suppose that we are looking at the graph of the function  $f(x) = c$ , and we decide to increase  $c$ . What happens to the graph of the function? The function is still constant, just with a larger constant, so the graph will be a horizontal line. Will it be higher or lower? Higher, because if the new constant, which we will call  $c'$ , is larger than  $c$ , then its graph will pass through the point  $(0, c')$ , which is higher than the point  $(0, c)$  on the  $y$ -axis. So increasing the constant in a constant function affects the graph of that function by moving it higher up the  $y$ -axis, while keeping it horizontal. What happens if we decrease  $c$ ? By the same argument (which you should be able to recreate now), the graph of the function will change by moving lower down the  $y$ -axis, while staying horizontal. You should understand now the affect of changing the parameter  $c$  on the graph of the function  $f(x) = c$ .

## Linear Functions

A linear function is a function of the form  $f(x) = mx + b$ , where  $m$  and  $b$  are constants. We call these functions linear because their graphs are lines in the plane.

Let us graph the function  $f(x) = 2x + 1$  to show why this is true. We begin by making a numerical table of values of  $f$ :

$x$	$f(x)$
-2	-3
-1	-1
0	1
1	3
2	5

Now let us plot these values as points on the plane. As we do this, we notice something interesting: every time the value of  $x$  increases by 1, the value of  $f(x)$  increases by 2. So, as we draw the graph of  $f(x)$ , all of the points lie along the same line. That line can be characterized by two values: first, how much  $f(x)$  increases as  $x$  increases by 1, which is called the slope, and second, the point at which the line intersects the  $y$ -axis, which is called the  $y$ -intercept. The slope of the graph of a linear function is given by  $m$ , and the  $y$ -intercept is given by  $b$ . In this case, the graph of  $f(x) = 2x + 1$  is the line with slope 2 which intersects the  $y$ -axis at the point  $(0, 1)$ .

Now, what would happen to the graph of a linear function  $f(x) = mx + b$  if we increase  $b$ , while keeping  $m$  constant? The new graph would have the same slope, but it would intersect the  $y$ -axis at a higher point.

If we decrease  $b$  while keeping  $m$  constant, again, the graph has the same slope, but now it intersects the  $y$ -axis at a lower point. Compare this to what happens when we increase or decrease  $c$  in constant functions.

What if we increase  $m$  while keeping  $b$  constant? The new line will intercept the  $y$ -axis at the same point, but now it will have a higher slope. You can imagine that, as we increase  $m$ , the line which is the graph of this function is pivoting on its  $y$ -intercept, turning counterclockwise around it. Likewise, if we decrease  $m$  while keeping  $b$  constant, the line will pivot on its  $y$ -intercept, turning clockwise around it.

Finally, we want to point out that, when  $m$  is 0, we simply have a constant function again. When  $m$  is positive, the line is sloped upward, and when  $m$  is negative, the line is sloped downward, that is, as  $x$  increases,  $f(x)$  decreases.

## Deriving Linear Functions

Next we learn how to derive the formula for a linear function when we know its value at two different values of  $x$ . To get the formula for a linear function, we need to know the values of  $m$  and  $b$ .

Suppose we are given the value of our linear function at  $x_1$  and  $x_2$ , that is, we already know what  $f(x_1)$  and  $f(x_2)$  are. We can compute the slope using the following formula:

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

There are a couple of things you need to remember about this formula: first, if you subtract the value of  $f$  at  $x_1$  from the value of  $f$  at  $x_2$ , then in the denominator, you have to subtract  $x_1$  from  $x_2$ . In other words, you have to keep the order in which you subtract points in the denominator the same as in the numerator. Second,  $f(x_1)$  and  $f(x_2)$  belong in the numerator. If they were in the denominator, and  $x_1$  and  $x_2$  were in the numerator, then if we tried to find the slope of a constant function, we would be dividing by 0. Try this so that you remember!

Let us do an example: suppose we know that  $f(x)$  is a linear function, and  $f(3) = 4$  and  $f(5) = -2$ . So  $x_1 = 3$ ,  $x_2 = 5$ ,  $f(x_1) = 4$ , and  $f(x_2) = -2$ . Applying the formula for slope, we get

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{-2 - 4}{5 - 3} = \frac{-6}{2} = -3.$$

Thus the slope of  $f$  is  $-3$ .

Here is another example: suppose  $g(t)$  is a linear function (we are changing notation here so that you become more comfortable using different letters), and we know that  $g(7) = 4$  and  $g(-2) = -23$ . Using  $t$  instead of  $x$ , we see that  $t_1 = 7$  and  $t_2 = -2$ , and using  $g$  instead of  $f$ , we get that  $g(t_1) = 4$  and  $g(t_2) = -23$ . We now apply the formula for slope, using  $t$  and  $g$  instead of  $x$  and  $f$ :

$$m = \frac{g(t_2) - g(t_1)}{t_2 - t_1} = \frac{-23 - 4}{-2 - 7} = \frac{-27}{-9} = 3.$$

Hence the slope of  $g(t)$  is 3.

There is one final point to be made about the formula for slope: if  $f(x)$  is linear, then it does not matter which two values of  $x$  you plug into the formula. Slope is a constant of a linear function: it does not change with  $x_1$  and  $x_2$ . If this is not clear to you, you should apply the slope formula to problem 5 in the homework due today. Try as many combinations of values of  $R$  as you can, and convince yourself that the slope will be the same, no matter what numbers you use for  $R_1$  and  $R_2$ .

Now that we know the value of  $m$ , we can derive the value of  $b$ . The way we do this is assume that  $f(x) = mx + b$ , and then plug in the value of  $x_1$  for  $x$ ,  $f(x_1)$  for  $f(x)$ , and the number we got from our formula for slope for  $m$ . We then solve for  $b$ .

Let us try out this method on the two examples we did before. First, we have  $f(x)$ , which is a linear function for which  $f(3) = 4$  and  $f(5) = -2$ . We know that the slope of  $f(x)$  is equal to  $-3$ . If we assume  $f(x) = mx + b$ , we get that

$$\begin{aligned} f(x_1) &= mx_1 + b \\ 4 &= -3 \cdot 3 + b \\ 4 &= -9 + b \\ 13 &= b. \end{aligned}$$

Therefore the formula for  $f$  is  $f(x) = -3x + 13$ . It is a good idea to check this formula by plugging in the second point we have, in this case  $f(5) = -2$ . Here we get that

$$\begin{aligned}f(x_2) &= -3x_2 + 13 \\-2 &= -3 \cdot 5 + 13 \\-2 &= -15 + 13 \\-2 &= 2,\end{aligned}$$

so the formula  $f(x) = -3x + 13$  is correct.

For our second example, we had that  $g(t)$  is a linear function such that  $g(7) = 4$  and  $g(-2) = -23$ . We applied the formula for slope we got that  $m = 3$ . Solving for  $b$ , we get that

$$\begin{aligned}g(t_1) &= mt_1 + b \\4 &= 3 \times 7 + b \\4 &= 21 + b \\-17 &= b.\end{aligned}$$

So our formula for  $g$  is  $g(t) = 3t - 17$ . We now check this formula with our second point:

$$\begin{aligned}g(t_2) &= 3t_2 - 17 \\-23 &= 3 \cdot -2 - 17 \\-23 &= -6 - 17 \\-23 &= -23.\end{aligned}$$

Thus we have confidence that our formula for  $g(t)$  is correct.

Given the formulas we just got in our two examples, you should be able to tell whether the graphs of these functions are sloped upward or downward or if they are flat, and whether they intersect the  $y$ -axis above, below, or at the origin. You should be able to derive a numerical table from these formulas, and you should be able to plot the graphs of these linear functions on a sheet of graph paper.