Implicit Functions

Defining Implicit Functions

Up until now in this course, we have only talked about functions, which assign to every real number x in their domain exactly one real number $f(x)$. The graphs of a function $f(x)$ is the set of all points (x, y) such that $y = f(x)$, and we usually visually the graph of a function as a curve for which every vertical line crosses that curve at most once. There are other curves that we can draw on the xy-plane which do not pass the vertical line test. One such curve is the circle of radius 1 centered at the origin. We can describe this circle with the relation

$$x^2 + y^2 = 1,$$

that is, the circle of radius 1 centered at the origin is the set of all points (x, y) such that $x^2 + y^2 = 1$. Now consider one point on this circle, the point $(0, 1)$. You may notice that if we remove some of the circle (for example, the lower half of the circle), the remaining curve is the graph of a function. This function, for which we will find a formula below, is called an implicit function, and finding implicit functions and, more importantly, finding the derivatives of implicit functions is the subject of today's lecture.

In general, we are interested in studying relations in which one function of x and y is equal to another function of x and y. A function f of x and y takes each ordered pair (x, y) and associates it to some number $f(x, y)$. A general way to write down the type of relations in which we are interested is:

$$f(x, y) = g(x, y).$$

The relation $x^2 + y^2 = 1$ which defines the circle of radius 1 centered at the origin is one such relation: in this case, $f(x, y) = x^2 + y^2$ and $g(x, y)$ is the constant function 1. Another such relation is $y - 1 = x^2 + 2x$. This relation also defines a curve, a parabola. How do we see this? The natural thing to do is to solve for y:

$$\begin{aligned} y - 1 &= x^2 + 2x \\ y &= x^2 + 2x + 1 \end{aligned}$$

Thus we see that the curve defined by the relation $y - 1 = x^2 + 2x$ is just the graph of a quadratic function. The function $y = x^2 + 2x + 1$ that we found by solving for y is called the implicit function of the relation $y - 1 = x^2 + 2x$. In general, any function we get by taking the relation $f(x, y) = g(x, y)$ and solving for y is called an implicit function for that relation. What complicates the situation is that a relation may have more than one implicit function.

The standard example of a relation of the form above which has more than one implicit function is, of course, $x^2 + y^2 = 1$. To see this, let us try to solve for y:

$$\begin{aligned} x^2 + y^2 &= 1 \\ y^2 &= 1 - x^2 \end{aligned}$$

Once we isolate y^2, we discover a problem: in order to get y from y^2, we need to take the square root of the right hand side, but we could the positive square root and get one implicit function, or we could take the negative square root and get another implicit function:

$$y = \sqrt{1 - x^2} \qquad \text{or} \qquad y = -\sqrt{1 - x^2}.$$

Both of these implicit functions has domain $-1 \leq x \leq 1$ (can you see why?). The graph of the first implicit function is the non-negative half of the circle, and the graph of the second is the non-positive half of the circle. Together, their graphs make the entire circle.

In general, a relation has multiple implicit functions if, while solving for y, we come to a step in which we have to make a choice, like a square root. Another possibility is given by the relation $\sin y = x$. Aside from the fact that we do not know how to get rid of the sine function like we would an exponent, we have the problem that, if $-1 \leq x \leq 1$, then there is an infinite number of numbers y such that $\sin y = x$, and each corresponds to an implicit function for this relation.

When a relation has multiple implicit functions, we tend to choose one of those implicit functions and study it alone instead of looking at all the implicit functions together. One way to do this is to choose a point (x, y) which satisfies the original relation (in other words, a point on the curve defined by the relation), and to take an implicit function $h(x)$ for which $y = h(x)$ (that is, an implicit function for which (x, y) is on the graph of that function). We call $h(x)$ the implicit function of the relation at the point (x, y). For example, we have the relation $x^2 + y^2 = 1$ and the point $(0, 1)$. This relation has two implicit functions, and only one of them, $y = \sqrt{1 - x^2}$, has the point $(0, 1)$ on its graph. This fits our intuitive idea of an implicit function from the introduction to this lecture, because the graph of this implicit function is the upper half of the circle. If, however, we took the point $(0, -1)$, then the implicit function of $x^2 + y^2 = 1$ at this point is $y = -\sqrt{1 - x^2}$.

Sometimes a point has more that one implicit function associated with it. For the relation $x^2 + y^2 = 1$, take the point $(1, 0)$. Both the implicit function $y = \sqrt{1 - x^2}$ and the implicit function $y = -\sqrt{1 - x^2}$ have the point $(1, 0)$ on their graphs. In a sense, a point which has more than one implicit function associated to it is a bad point for the relation. Can you find another bad point for the relation $x^2 + y^2 = 1$? There is a way to identify bad points for a relation, which we will see when we learn how to differentiate implicit functions.

The Conic Sections

In this section, we will go through a series of examples of relations with which all calculus students should be familiar: the conic sections. A conic section is a relation of the form

$$ax^2 + bxy + cy^2 + px + qy + r = 0,$$

where a, b, c, p, q, and r are all constants. We call these relations conic sections because the shape of the curves corresponding to these relations can take one of three forms: an ellipse, a parabola, or a hyperbola. These three shapes are precisely the curves we get when we intersect a double cone with a plane.

- An ellipse is a type of elongated circle (a circle is one example of an ellipse). For example, the relation

  $$\frac{(x - h)^2}{A^2} + \frac{(y - k)^2}{B^2} = 1,$$

  which can be expanded and rewritten into the form above, gives us an ellipse centered at the point (h, k) with width $2A$ and height $2B$. In particular, the relation

  $$(x - h)^2 + (y - k)^2 = R^2$$

  gives us a circle centered at (h, k) of radius R. The curve of a conic section is an ellipse when $b^2 - 4ac < 0$.

- We already know what the shape of a parabola is. The curve of a conic section is a parabola when $b^2 - 4ac = 0$. Here we allow for the parabola to be open in any direction, including horizontally. So, for example, the conic section $x - y^2 + 3x - 2 = 0$ gives us a parabola which is open to the right.

- A hyperbola, roughly speaking, is a curve which consists of two disconnected parabola-like curves which are open in opposite directions. A good example of a hyperbola is the graph of the function $y = x^{-1}$, which we can rewrite into the form $xy = 1$ (making it a conic section). Another example is the curve of the relation $y^2 - x^2 = 1$. A conic section gives a hyperbola when $b^2 - 4ac > 0$.

Let us now do a couple of examples of finding the implicit functions of conic section. First, let us take

$$x^2 + 2xy + y^2 - y + x = 0.$$

In this case, $a = 1$, $b = 2$, and $c = 1$, so $b^2 - 4ac = 0$ and the curve of this relation is a parabola. Let us find its implicit functions. A conic section will have at most two implicit function. We find them by treating this relation as if it is a quadratic function of y, and that x is just part of the coefficients:

$$\begin{aligned} x^2 + 2xy + y^2 - y + x &= 0 \\ y^2 + (2x - 1)y + (x^2 + x) &= 0. \end{aligned}$$

Now we just solve for y using the quadratic formula:

$$\begin{aligned} y &= \frac{-(2x-1)+\sqrt{(2x-1)^2-4\cdot 1\cdot (x^2+x)}}{2\cdot 1} & y &= \frac{-(2x-1)-\sqrt{(2x-1)^2-4\cdot 1\cdot (x^2+x)}}{2\cdot 1} \\ y &= \frac{-(2x-1)+\sqrt{4x^2-4x+1-4x^2-4x}}{2} & y &= \frac{-(2x-1)-\sqrt{4x^2-4x+1-4x^2-4x}}{2} \\ y &= \frac{-(2x-1)+\sqrt{1-8x}}{2} & y &= \frac{-(2x-1)-\sqrt{1-8x}}{2}. \end{aligned}$$

For instance, the first function above gives us the implicit function of this relation at the point $(-3,6)$, since

$$y = \frac{-(2\cdot 3-1)+\sqrt{1-8\cdot 3}}{2} = \frac{-(-7)+\sqrt{25}}{2} = \frac{7+5}{2} = \frac{12}{2} = 6.$$

For our second example, take

$$4y^2 - 9x^2 = 1.$$

Here we have that $a=-9$, $b=0$, and $c=4$, so $b^2-4ac = 0^2-4\cdot(-9)\cdot 4 = 144$, making the curve of this conic section a hyperbola. Finding the two implicit functions of this relation is reasonably easy:

$$\begin{aligned} 4y^2 - 9x^2 &= 1 \\ 4y^2 &= 1+9x^2 \\ y^2 &= \frac{1+9x^2}{4} \\ y = \frac{\sqrt{1+9x^2}}{2} \quad \text{or} \quad y &= -\frac{\sqrt{1+9x^2}}{2}. \end{aligned}$$

Notice that in the first implicit function, y is always positive, while in the second implicit function, y is always negative, so, if you know that $(2,3)$ and $(-2,-3)$ are two points which satisfy this relation (check this!) you know instantly that $y=\frac{\sqrt{1+9x^2}}{2}$ is the implicit function at $(2,3)$ and $y=-\frac{\sqrt{1+9x^2}}{2}$ is the implicit function at $(-2,-3)$. Try sketching the graphs of these two implicit functions to get a better idea of what a hyperbola is supposed to look like. Do you see any symmetries?

Implicit Differentiation

Given all this work trying to find implicit functions, it may surprise you to know that it is not necessary to find the formula for an implicit function in order to find its derivative. Indeed, sometimes it is impossible to find the formula for an implicit function without having to make some new type of function in the process: consider again the relation $\sin y = x$ for an example of this. Nevertheless, we can find the derivative of the implicit functions of this relation, where the derivative exists, using a process called implicit differentiation.

The idea behind implicit differentiation is to treat y as a function of x (which is what we are trying to do anyway). To emphasize this, let us rewrite the relation above, replacing y with $y(x)$:

$$\sin(y(x)) = x.$$

Now we differentiate each side of this equation, and set their derivatives equal to each other. Since we do not know the formula for $y(x)$, we just leave its derivative as $y'(x)$:

$$\cos(y(x)) \cdot y'(x) = 1.$$

Finally, we solve for $y'(x)$ to get its formula:

$$y'(x) = \frac{1}{\cos(y(x))} = \frac{1}{\cos y}.$$

At the end we turn $y(x)$ back into y to make the notation less cumbersome. What this tells us is that, even though we do not have formulae for the implicit functions of $\sin y = x$, we know that the derivative of those implicit functions is given by $\frac{1}{\cos x}$. So, for example, we know that the point $\left(\frac{1}{2}, \frac{\pi}{6}\right)$ is a point on the curve of the relation $\sin y = x$:

$$\sin y = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2} = x.$$

Whatever the formula for the implicit function of $\sin y = x$ at $\left(\frac{1}{2}, \frac{\pi}{6}\right)$ is, we know that its derivative at that point is

$$y'(x) = \frac{1}{\cos\left(\frac{\pi}{6}\right)} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}.$$

We need to analyze the relation $\sin y = x$ and its derivative a bit more, but first let us go back to our first example, the relation $x^2 + y^2 = 1$. In this case, we know how to find the implicit functions of this relation, but let us use implicit differentiation anyway. First, we rewrite this relation by replacing y with $y(x)$:

$$x^2 + (y(x))^2 = 1.$$

Next, we differentiate both sides and set the derivatives equal to each other:

$$2x + 2y(x) \cdot y'(x) = 0.$$

Finally we solve for $y'(x)$:

$$\begin{aligned} 2x + 2y(x) \cdot y'(x) &= 0 \\ 2y(x) \cdot y'(x) &= -2x \\ y'(x) &= \frac{-2x}{2y(x)} \\ y'(x) &= -\frac{x}{y}. \end{aligned}$$

Before we said that one point on this circle is $(0, 1)$. This formula for the derivative of the implicit function tells that the slope of the curve at that point should be 0, and this is precisely what we see when draw out the circle $x^2 + y^2 = 1$ and plot the point $(0, 1)$. Another point on this circle is $(1, 0)$. We considered this a bad point, because it had more than one implicit function associated to it. When we try to apply our formula for $y'(x)$ to this point, we see that the formula is undefined at $(1, 0)$. This makes sense: the tangent line to the circle $x^2 + y^2 = 1$ at $(1, 0)$ is a vertical line, so its slope is undefined. What happened here is that the formula for $y'(x)$ is a quotient, and at $(1, 0)$, the denominator is 0. This gives us a way to find the bad points of a relation, the points where the implicit function for the relation is not well-defined: a bad point is a point where the formula for $y'(x)$ is undefined. In this case, the bad points for the relation $x^2 + y^2 = 1$ are the points on the circle where $y = 0$, so that

$$\begin{aligned} x^2 + y^2 &= 1 \\ x^2 + 0^2 &= 1 \\ x^2 &= 1 \\ x = 1 \quad \text{or} &\quad x = -1. \end{aligned}$$

So the bad points are $(1, 0)$ and $(-1, 0)$, which is exactly what we would expect looking at the circle or radius 1 centered at the origin.

Going back to the relation $\sin y = x$, let us try to find the bad points of this relation. The formula for the derivative that we got before is

$$y'(x) = \frac{1}{\cos y},$$

so the bad points will be where $\cos y = 0$. So the bad points have y-coordinates

$$\ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots.$$

Since $x = \sin y$ by the formula for the relation, we get that the bad points of $\sin y = x$ are the points

$$\ldots, \left(-\frac{3\pi}{2}, 1\right), \left(-\frac{\pi}{2}, -1\right), \left(\frac{\pi}{2}, 1\right), \left(\frac{3\pi}{2}, -1\right), \ldots.$$

Let us do one more example of implicit differentiation using the conic section

$$x^2 + 2xy + y^2 - y + x = 0.$$

Replacing y with $y(x)$, we get

$$x^2 + 2x \cdot y(x) + (y(x))^2 - y(x) + x = 0.$$

Differentiating both sides and setting the derivatives equal to each other, we have that

$$2x + 2y(x) + 2x \cdot y'(x) + 2y(x) \cdot y'(x) - y'(x) + 1 = 0.$$

Now we solve for $y'(x)$:

$$\begin{aligned}
2x + 2y(x) + 2x \cdot y'(x) + 2y(x) \cdot y'(x) - y'(x) + 1 = 0& \\
(2x + 2y(x) + 1) + (2x + 2y(x) - 1) \cdot y'(x) = 0& \\
(2x + 2y(x) - 1) \cdot y'(x) &= -(2x + 2y(x) + 1) \\
y'(x) &= \frac{-(2x + 2y(x) + 1)}{2x + 2y(x) - 1} \\
y'(x) &= -\frac{2x + 2y + 1}{2x + 2y - 1}.
\end{aligned}$$

So, for example, the derivative of the implicit function of this conic section at the point $(-3, 6)$ is

$$y'(x) = -\frac{2x + 2y + 1}{2x + 2y - 1} = -\frac{2 \cdot (-3) + 2 \cdot 6 + 1}{2 \cdot (-3) + 2 \cdot 6 - 1} = -\frac{-6 + 12 + 1}{-6 + 12 - 1} = -\frac{7}{5}.$$

Try differentiating the formula for the implicit function that we got before to verify this value for the derivative at $(-3, 6)$. The bad points of this conic section are when $2x + 2y - 1 = 0$. If we were being industrious, we could solve for the bad points explicitly, but let us leave that alone for now.