The Notion of a Derivative and Cubic Functions

Derivatives in General

In our last lecture, we talked about the derivative of a quadratic function. We said that the derivative of a quadratic function at a point is the slope of the tangent line to the graph of that function at that point.

We can extend this notion of derivative to many other functions. Let us draw the graph of a function $f(x)$ on an xy-plane. Make sure that there are no breaks or sharp corners in the graph. Pick a point on the graph somewhere in the middle (in other words, do not pick an endpoint). Draw a dotted vertical line down to the x-axis, and mark the x value of that point as $x = p$. Therefore the point on the graph corresponding to $x = p$ is $(p, f(p))$

We are going to graphically illustrate how to find the derivative of $f(x)$ at $x = p$. To do this, we mark the x-axis a little to the right of $x = p$, calling this point $x = p + h$. The letter h represents the distance between the point on the x-axis where $x = p$ and our new point at $x = p + h$. We then draw a dotted line up from $x = p + h$ to the graph of $f(x)$ and mark that point on the graph. The coordinates of that point are $(p + h, f(p + h))$. Now draw the line connecting the points $(p, f(p))$ and $(p + h, f(p + h))$. What is the slope of this line? We use the formula for slope and the two points on the line that we have to find that

$$m = \frac{f(p+h) - f(p)}{(p+h) - p} = \frac{f(p+h) - f(p)}{h}.$$

The line we drew passing through the points $(p, f(p))$ and $(p + h, f(p + h))$ is an approximation of the tangent line to the graph of $f(x)$ at $x = p$. The slope of that line is an approximation for the derivative of $f(x)$ at $x = p$, which is the slope of the tangent line. How do we get a better approximation? We can make h a smaller number. So let us pick a point on the x-axis somewhere in between $x = p$ and $x = p + h$. Mark this point as $x = p + h'$. Since $x = p + h'$ is in between $x = p$ and $x = p + h$, it must be true that $h' < h$. Now let us, again, draw a dotted vertical line up from the x-axis to the graph of $f(x)$ and mark the point where the vertical line touches the graph. This point has coordinates $(p + h', f(p + h'))$. We can draw the line that connects $(p, f(p))$ and $(p + h', f(p + h'))$, and this line is now a better approximation for the tangent line to the graph of $f(x)$ at $x = p$. We also want a better approximation for the derivative of $f(x)$ at $x = p$, and to do that we find the slope of the line connecting $(p, f(p))$ with $(p + h', f(p + h'))$:

$$m = \frac{f(p+h') - f(p)}{(p+h') - p} = \frac{f(p+h') - f(p)}{h'}.$$

We can repeat this process over and over again, until we get we find a value for h which is small enough that our approximation for the tangent line to the graph of $f(x)$ at $x = p$, and for the derivative of $f(x)$ at $x = p$, are close enough for whatever purposes we have. There is a way to write this notion of approximation:

$$\frac{\mathrm{d}f}{\mathrm{d}x}(p) = \lim_{h \to 0} \frac{f(p+h) - f(p)}{h}.$$

We read this equation as "the derivative of $f(x)$ at $x = p$ is equal to the limit as h goes to 0 of $f(p+h) - f(p)$ divided by h." This is a fancy way to writing the idea that we can approximate the derivative of the function at $x = p$ using the slope of the line connecting $(p, f(p))$ and $(p + h, f(p + h))$ by making h smaller and smaller.

You do not need to know how to use the limit precisely yet. All you need to know is that the derivative is useful for finding the tangent line to many functions, not just quadratic functions; that you can approximate the derivative using the technique we just described; and that there is a precise idea of approximation, called the limit, which we will study later.

Cubic Functions

We now move on to a more concrete class of functions: the cubic functions. A cubic function is a function of the form $f(x) = ax^3 + bx^2 + cx + d$, where a, b, c, and d are constants, and $a \neq 0$.

When we studied quadratic functions, we described a method to find the roots of a quadratic function: we find the values of x for which a quadratic function equals 0 by using the quadratic formula. There does exist a formula for finding the roots of a cubic function, but this formula is very complicated, and not very useful to us. So, for many cubic functions, we will not be able to find the roots of those functions in this class.

The graphs of cubic functions come in three basic forms. They can be illustrated by plotting the graphs of the functions $f(x) = x^3$, $g(x) = x^3 + 3x$, and $h(x) = x^3 - 3x$:

x	$f(x)$
-2	-8
-1	-1
0	0
1	1
2	8

x	$g(x)$
-2	-14
-1	-4
0	0
1	4
2	14

x	$h(x)$
-2	-2
-1	2
0	0
1	-2
2	2

The first function $f(x) = x^3$, has no maximum or minimum, and the slope of the function is flat at $x = 0$. The second function $g(x) = x^3 + 3x$, also has no maximum and no minimum, but its graph is much steeper than that of $f(x)$, and the slope of the function at $x = 0$ is positive. Finally, $h(x) = x^3 - x$ is quite different from the other two: it has what are called a local maximum and a local minimum, points which are the maximum and minimum for the function for all the values of x near that point, but not necessarily for the function as a whole. For this function, the local maximum is at $x = -1$, and the local minimum is at $x = 1$. We will learn how to find those points later in the course. We also note that the graph of $h(x)$ has negative slope at $x = 0$.

All of these characterizations of the graphs of cubic functions lead to the following question: what is the derivative of a cubic function?

Derivatives of Cubic Functions

Suppose that $f(x) = ax^3 + bx^2 + cx + d$, and we want to find the derivative of $f(x)$ at the point $x = p$. The formula for the derivative of $f(x)$ at $x = p$ is

$$\frac{\mathrm{d}f}{\mathrm{d}x}(p) = 3ap^2 + 2bp + c.$$

Do you see any similarities between the formula for the derivative of a quadratic function and that of a cubic function? Soon, you will see that there is a method to find the formula of any polynomial, no matter how complicated it might be.

Let us do an example: we said earlier that the slope of the function $g(x) = x^3 + 3x$ is positive at $x = 0$. To be precise, what we are really saying is that the slope of the tangent line to the graph of $g(x)$ at $x = 0$ is positive. Let us find the value of that slope. We can rewrite the formula for $g(x)$ as

$$g(x) = 1x^3 + 0x^2 + 3x + 0.$$

So, to apply our formula for the derivative, we use $a = 1$, $b = 0$, $c = 3$, and for the point at which we are finding the derivative, $p = 0$. We get then that

$$\frac{\mathrm{d}g}{\mathrm{d}x}(0) = 3ap^2 + 2bp + c = 3 \cdot 1 \cdot 0^2 + 2 \cdot 0 \cdot 0 + 3 = 3.$$

So the slope of the tangent line to the graph of the function $g(x)$ at $x = 0$ is 3. We know that $g(0) = 0$, so the y-intercept of that tangent line is also 0. Thus the equation for the tangent line to the graph of $g(x)$ at $x = 0$ is $y = 3x$.

Let us also find the derivative of the function $h(x) = x^3 - 3x$ at $x = -1$, which is supposed to be a local maximum of the function. Applying the formula for the derivative, using $a = 1$, $b = 0$, $c = -3$, and $p = -1$, we get that

$$\frac{\mathrm{d}h}{\mathrm{d}x}(-1) = 3ap^2 + 2bp + c = 3 \cdot 1 \cdot (-1)^2 + 2 \cdot 0 \cdot (-1) - 3 = 3 + 0 - 3 = 0.$$

Is this the value of the derivative of $h(x)$ at $x = -1$ that you expected, considering that we believed that $x = -1$ was a local maximum?