# Graphing Derivatives

## Sketching the Graphs of Derivatives

Today we are going to learn how to sketch the graph of the derivative function. Recall from the last lecture that, for many functions, including all polynomials, we can think of the derivative not just at one point along the graph, but as a function which assigns to every point the slope of the tangent line to the graph of the function at that point. Since the derivative is a function, we can graph it on the $xy$-plane just like any other function.

For example, consider the constant function $f(x) = 3$. The derivative of any constant function at any point is 0, so the derivative of $f(x)$ is the constant function 0. If we were to graph the derivative, the curve of the graph would be the $x$-axis, which is the line $y = 0$.

As another example, let $g(x) = -2x + 5$. This is a linear function, and we recall that the derivative of any linear function at any point is the slope of that linear function. Therefore $\frac{\mathrm{d}g}{\mathrm{d}x}(x)$ is the constant function $-2$. If we sketch the graph of $\frac{\mathrm{d}g}{\mathrm{d}x}(x)$, we get the horizontal line $y = -2$.

So far, we have sketched the derivatives of constant and linear functions. In both cases, the derivatives were constant functions, which means that they are not very interesting. Now let us study the graph of a quadratic function. Let $h(x) = x^2 - 4x + 3$. The derivative of $h(x)$ is $\frac{\mathrm{d}h}{\mathrm{d}x}(x) = 2x - 4$. So the derivative of $h(x)$ is a linear function, the slope of which is 2 and the $y$-intercept of which is the point $(0, 4)$. The $x$-intercept of $\frac{\mathrm{d}h}{\mathrm{d}x}$, which is the point where the derivative of $h(x)$ is 0, is the point $(2, 0)$. Now let us superimpose on the graph of the derivative of $h(x)$ the graph of $h(x)$ itself. We sketch the parabola, and we find that the minimum of $h(x)$, the point where we anticipate the derivative of $h(x)$ to be 0, is precisely at $x = 2$, where the line which is the graph of the derivative crosses the $x$-axis. This gives us one clue as to how to graph of the derivative of an unknown function if you know what the graph of the function looks like: find the points on the graph of the function where the derivative appears to be zero. These points, which we call critical points, are precisely where the graph of the derivative will touch the $x$-axis.

Next, let us graph the derivative of a cubic function. Let $k(x) = x^3 - 3x$, which is a cubic function we have graphed before. Graph this function: you should get the familiar curve of a cubic, with a local maximum at $x = -1$ and a local minimum at $x = 1$. These are the only critical points of $k(x)$, the points where the derivative is 0, as we can see by examining the graph of $k(x)$. Now, we know how to find the derivative of $k(x)$, but before we do let us try to sketch the derivative function just from the information given by the graph of $k(x)$. The first thing we can do is use the critical points. We know that at the critical points, the graph of the derivative of $k(x)$ touches the $x$-axis. So we mark two points on the $xy$-plane, $(-1, 0)$ and $(1, 0)$. We know that the graph of the derivative will run through these two points.

The next step is to find out where the derivative is positive and where it is negative. Studying the graph of $k(x) = x^3 - 3x$, we see that the derivative is positive when $x > 1$ and when $x < -1$, and it is negative when $x$ is between $-1$ and 1.

Now we need to find the behavior of the derivative on the intervals where it is positive or negative. We need to find out where the derivative reaches its maximum, where does it reach its minimum, where it is increasing, and where it is decreasing. First, let us study the interval $-1 \leq x \leq 1$. Here, we see that the derivative begins at a value of 0 at $x = -1$ and becomes more and more negative, until it reaches $x = 0$, where it reaches its minimum. This is precisely where the graph stops looking like it is concave down (like a parabola open downward) and starts looking like it is concave up (like a parabola open upward). We call this point an inflection point of $f(x)$. We estimate the derivative at the inflection point, this minimum of the derivative: it is somewhere around $-3$. The derivative then begins to increase (although it is still negative), until it gets back to 0 again at $x = 1$.

How do we use this information to sketch the derivative? We first mark the point where the derivative reaches its minimum on the graph for the derivative. This minimum is at $x = 0$, the inflection point, and we estimated the derivative of $k(x)$ at $x = 0$ to be around $-3$, so we mark the point $(0, -3)$ on the $xy$-plane where we are sketching the graph of the derivative, because we know that the graph of the derivative will pass through this point. We also know that this is a local minimum for the derivative, so we know that the slope of the derivative function at $x = 0$, which we call the second derivative, is 0 at this point. How should our sketch reflect this fact? As we sketch out the graph of the derivative of $k(x)$ between $x = -1$ and $x = 1$, we connect the three points we have on the graph: $(-1, 0)$, $(0, -3)$, and $(1, 0)$. We begin at $(-1, 0)$ and, since the derivative is negative between $x = -1$ and $x = 1$, we draw our graph down below the $x$-axis. We keep drawing downward, but as begin to approach the point $(0, -3)$, we flatten out the curve we are drawing, so that when we finally touch the point $(0, -3)$, the slope of the curve is 0. This is the minimum of the graph of the derivative between $x = -1$ and $x = 1$, so from $(0, -3)$ to $(1, 0)$, the curve is going to rise, first slowly, but as it gets closer to $(1, 0)$, faster and faster. We finally touch the point $(1, 0)$, and now we have a sketch of the derivative of $k(x)$ between $x = -1$ and $x = 1$.

What about the region on the graph left of $x = -1$? We look at the graph of $k(x)$ in this region, and we see that the derivative is near zero, but positive, just to the left of $x = -1$, and as the graph moves farther left, the derivative becomes more and more positive: the derivative never reaches a maximum. We can draw this into our sketch of the derivative of $k(x)$ by starting the sketch at $(-1, 0)$ and continuing the sketch that we already made between $x = -1$ and $x = 1$ so that our sketch is always positive and always increasing as we move to the left. We do the same for the region of the graph to the right of $x = 1$, and we get a similar result: to the right of $x = 1$, our sketch of the graph of the derivative of $k(x)$ is always positive and always increasing as we move to the right. The resulting sketch looks like a parabola open upward. Does this make sense? Yes, because the derivative function of $k(x)$ is $\frac{\mathrm{d}k}{\mathrm{d}x}(x) = 3x^2 - 3$. Our sketch is accurate.

So, given the graph of an unknown function $f(x)$ which has no gaps or corners in it, we can now sketch its derivative $\frac{\mathrm{d}f}{\mathrm{d}x}(x)$. Here are some steps you should follow:

1. Begin by locating the critical points of the function, which, as we learned today, are the $x$-values where the derivative of the function (the slope of the tangent line at that point) are zero. On the $xy$-plane on which you will draw your sketch of the derivative, mark those $x$-values on the $x$-axis. We know that the derivative function will touch the $x$-axis at those points.

2. Next, examine the graph of the function between the critical points (as well as to the left of the first critical point and to the right of the last critical point). Decide whether the derivative of the function is positive or negative between two adjacent critical points: it can be one or the other, never both. Find the places between the critical points where the derivative changes from concave up to concave down and vice versa: these are the inflection points. There is usually at least one inflection point between two critical points, and maybe more. Estimate the value of the derivative at these points, and mark those points on the graph (with the $y$-value of the point being your estimate for the derivative).

3. Finally, make your sketch of the graph of the derivative by connecting the points that you marked on your $xy$-plane. Make sure to make your graph is smooth (no gaps or corners) and make sure that, at the points you marked during the second step, that the slope of your sketch is 0.